Problems in particle dynamics

Questions

1. A particle moves in a straight line under a force towards a fixed point O on the straight line. The magnitude of the force is equal to $\frac{\mu}{x^2} - \frac{\lambda}{x^3}$ when the particle is at a distance x from O. It starts from rest at a distance a from the fixed point, show that it oscillates between this distance and the distance $\frac{\lambda a}{2\mu a - \lambda}$ and that its periodic time is $\frac{2\pi \mu a^3}{(2\mu a-\lambda)^{\frac{3}{2}}}$.

   Solution: Let v be the velocity of the particle at a distance x from the fixed point on the line.

   The equation of motion of the particle is

$$v\frac{dv}{dx} = -\left(\frac{\mu}{x^2} - \frac{\lambda}{x^3}\right) \tag{1}$$

$$\therefore vdv = \left(-\frac{\mu}{x^2} + \frac{\lambda}{x^3}\right) dx$$

   Integrating, we get, $\frac{v^2}{2} = \frac{\mu}{x} - \frac{\lambda}{2x^2} + C$, where C is the constant of integration.
   Initially, when $x = a$, $v = 0$. So, $C = \frac{\lambda}{2a^2} - \frac{\mu}{a}$.

$$\therefore v^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right) - \lambda\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \left(\frac{1}{x} - \frac{1}{a}\right)\left\{2\mu - \lambda\left(\frac{1}{x} + \frac{1}{a}\right)\right\} \tag{2}$$

   $\therefore v = 0$, when either $2\mu - \lambda\left(\frac{1}{x} + \frac{1}{a}\right) = 0$ or $\left(\frac{1}{x} - \frac{1}{a}\right) = 0$ i.e. $x = \frac{\lambda a}{2\mu a - \lambda}$ and $x = a$.
   This shows that the particle's motion under the said attraction towards a fixed point in the line is oscillatory between the distances $x = a$ and $x = \frac{\lambda a}{2\mu a - \lambda} = \alpha a$ (say).
   From equation (2), $v = -\sqrt{\left(\frac{1}{x} - \frac{1}{a}\right)\left\{2\mu - \lambda\left(\frac{1}{x} + \frac{1}{a}\right)\right\}}$,
   the -ve sign is taken because x decreases as t increases.
   So, $v = -\sqrt{\frac{a-x}{ax}\left((2\mu - \frac{\lambda}{a}) - \frac{\lambda}{x}\right)} = -\sqrt{\frac{a-x}{ax}\left(\frac{2\mu a - \lambda}{a} - \frac{\lambda}{x}\right)}$

$$= -\sqrt{\frac{a-x}{ax}\lambda\left(\frac{2\mu a - \lambda}{\lambda a} - \frac{1}{x}\right)} = -\sqrt{\frac{a-x}{ax}\lambda\left(\frac{1}{\alpha a} - \frac{1}{x}\right)} = -\frac{\sqrt{\frac{\lambda}{\alpha}(a-x)(x-\alpha a)}}{ax}.$$

   Let T be the periodic time.

$$\int_a^{\alpha a} \frac{axdx}{\sqrt{\frac{\lambda}{\alpha}(a-x)(x-\alpha a)}} = -\int_0^{T/2} dt,$$

$$T = -2a\sqrt{\frac{\alpha}{\lambda}}\int_a^{\alpha a}\frac{xdx}{\sqrt{(a-x)(x-\alpha a)}} = 4a^2\sqrt{\frac{\alpha}{\lambda}}\int_0^{\pi/2}\left(\sin^2\theta + \alpha\cos^2\theta\right)d\theta$$

[putting $x = a\sin^2\theta + \alpha a\cos^2\theta$, so $dx = 2(a - a\alpha)\sin\theta\cos\theta d\theta$ and $(a - x) = (a - a\alpha)\cos^2\theta$, $(x - a\alpha) = (a - a\alpha)\sin^2\theta$; Also $x = a$ when $\theta = \pi/2$, $x = a\alpha$ when $\theta = 0$]

$$= 4a^2\sqrt{\frac{\alpha}{\lambda}}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2} + \alpha\frac{1+\cos 2\theta}{2}\right)d\theta = 4a^2\sqrt{\frac{\alpha}{\lambda}}\int_0^{\pi/2}\left(\frac{1+\alpha}{2} + \frac{(\alpha-1)\cos 2\theta}{2}\right)d\theta$$

$$= 4a^2\sqrt{\frac{\alpha}{\lambda}}\frac{(1+\alpha)\pi}{2} = \frac{2\pi\mu a^3}{(2\mu a - \lambda)^{\frac{3}{2}}}$$ [Substituting the value of $\alpha = \frac{\lambda}{2\mu a - \lambda}$, $1 + \alpha = \frac{2\mu a}{2\mu a - \lambda}$]

2. A particle moves along X-axis with an acceleration $\frac{\mu}{x^3}$, where $\mu (> 0)$ is constant and x is the distance from origin, If it starts from rest at $x = a$ and the acceleration is towards the origin. Find the time taken to reach $x = a/4$ from $x = 3a/4$.

   [Hints: The equation of motion of the particle is $v\frac{dv}{dx} = -\frac{\mu}{x^3}$

   Integrating, $\frac{v^2}{2} = \frac{\mu}{2x^2} + C_1$

   Initially at $x = a$, $v = 0$. $\therefore C_1 = -\frac{\mu}{2a^2}$

   $v^2 = \mu\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \mu\frac{a^2 - x^2}{a^2x^2}$

   $v = -\sqrt{\mu}\frac{\sqrt{a^2 - x^2}}{ax}$

   or, $\frac{xdx}{\sqrt{a^2 - x^2}} = -\frac{\sqrt{\mu}}{a}dt$

   $\int_{3a/4}^{a/4}\frac{xdx}{\sqrt{a^2 - x^2}} = -\int_0^T\frac{\sqrt{\mu}}{a}dt$]

3. A particle moves along X axis under the acceleration $\frac{\mu}{x^2}$ towards the origin, where μ is a positive constant. Find the time required by the particle in describing the path from $x = 3a/4$ to $x = a/4$ if the particle starts from rest at $x = a$. Also show that this time is one-third the time from $x = a$ to $x = 0$.

   [**Try yourself**]

   [Answer: $\frac{a^{3/2}}{\sqrt{2\mu}}\frac{\pi}{6}$]

4. A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Show that the time it takes in reaching a height h is $\frac{1}{3}\sqrt{\frac{2R}{g}}\left\{\left(1 + \frac{h}{R}\right)^{\frac{3}{2}} - 1\right\}$, where R is the radius of the earth.

   Solution: We know that the earth attracts a particle outside its surface with a force varying inversely as the square of the distance of the particle from its centre and also attracts a particle inside its surface with a force varying directly as its distance from its centre, it being supposed that the earth is spherical.

Given that R is the radius of the earth.

If x be the distance of the particle from the earth's centre and v be its velocity at any time t, after it is projected from a point on the surface, then the acceleration of the particle is given by $v\dfrac{dv}{dx} = -\dfrac{\mu}{x^2}$ (1)

where $\dfrac{\mu}{R^2} = g =$the acceleration due to gravity on the earth's surface

Writing this equation as $v\dfrac{dv}{dx} = -\dfrac{gR^2}{x^2}$

Let us assume u is the velocity of projection of the particle just sufficient to carry it to infinity.

$\therefore \displaystyle\int_u^0 vdv = \int_R^\infty -\frac{gR^2}{x^2}dx$ or, $\dfrac{u^2}{2} = \dfrac{gR^2}{R}$ i.e. $u^2 = 2gR$ or, $u = \sqrt{2gR}$

Now, we have $\displaystyle\int_{\sqrt{2gR}}^v vdv = \int_R^x -\frac{gR^2}{x^2}dx$

$\therefore v^2 - 2gR = 2gR^2\left(\dfrac{1}{x} - \dfrac{1}{R}\right)$ i.e. $v^2 = \dfrac{2gR^2}{x}$ or, $v = R\sqrt{\dfrac{2g}{x}}$, the positive sign is taken since the velocity is in the direction of increasing x.

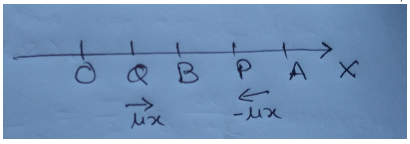
$\dfrac{dx}{dt} = R\sqrt{\dfrac{2g}{x}}$

Let T be the time required to reach a height h above the surface of the earth, then we have $\displaystyle\int_R^{R+h} \sqrt{x}dx = \int_0^T R\sqrt{2g}dt$ or, $R\sqrt{2g}T = \dfrac{2}{3}\left((R+h)^{3/2} - R^{3/2}\right) = \dfrac{2}{3}R^{3/2}\left((1+\dfrac{h}{R})^{3/2} - 1\right)$.

Therefore, $T = \frac{1}{3}\sqrt{\frac{2R}{g}}\left(\left(1+\dfrac{h}{R}\right)^{\frac{3}{2}} - 1\right)$

5. A particle moves along the axis of X starting from rest at $x = a$; for an interval t_1 from the beginning of the motion the acceleration is $(-\mu x)$, for a subsequent time t_2 the acceleration is μx and at the end of this interval the particle is at the origin again. Prove that $\tan(\sqrt{\mu}t_1)\tanh(\sqrt{\mu}t_2) = 1$.

Solution: Let the particle starts from rest at A and come to B in an interval of time t_1 under the attraction $-\mu x$. Let P be the position of the particle at time t between A and B,



where OP=x.

Then the equation of motion of the particle is $v\dfrac{dv}{dx} = -\mu x$ (1)

Integrating, we get, $\dfrac{v^2}{2} = -\dfrac{\mu x^2}{2} + C_1$, where, C_1 is the constant of integration.

Initially at $x = a$, $v = 0$. $\therefore C_1 = \dfrac{\mu a^2}{2}$. So, $v^2 = \mu\left(a^2 - x^2\right)$.

$v = -\sqrt{\mu\left(a^2 - x^2\right)}$, (2)

the -ve sign is taken because x decreases as t increases.

So, $\dfrac{dx}{dt} = -\sqrt{\mu\,(a^2 - x^2)}$ or, $\dfrac{-dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}dt$

Integrating, we have $\cos^{-1}\dfrac{x}{a} = \sqrt{\mu}t + D_1$, where, D_1 is the constant of integration.

Initially $x = a$ when $t = 0$, $\therefore D_1 = 0$.

Consequently, $x = a\text{cos}(\sqrt{\mu}t)$ So, $v = \dot{x} = -a\sqrt{\mu}\text{sin}(\sqrt{\mu}t)$

Let us assume $OB = b$ and V be the velocity at B when $t = t_1$

So, at $x = b$, $b = a\text{cos}(\sqrt{\mu}t_1)$ and $V = -a\sqrt{\mu}\text{sin}(\sqrt{\mu}t_1)$.

Thus the particle will cross the point B and will move towards the origin O. But as the particle cross the point B, the acceleration of the particle becomes μx at any point Q between B and O where $OQ = x$. Therefore, the equation of motion of the particle from B to O is

$$\frac{d^2x}{dt^2} = \mu x \textit{ i.e. } (D^2 - \mu)x = 0 \tag{3}$$

where $D \equiv \dfrac{d}{dt}$

The general solution is $x = C_2\text{cosh}(\sqrt{\mu}t) + D_2\text{sinh}(\sqrt{\mu}t)$, where C_2 and D_2 are constants of integration.

Thus, $\dot{x} = \sqrt{\mu}\left(C_2\text{sinh}(\sqrt{\mu}t) + D_2\text{cosh}(\sqrt{\mu}t)\right)$

For motion from B to O, at B we have $x = b$ and $\dot{x} = V$ when $t = 0$

$\therefore C_2 = b$ and $D_2 = V/\sqrt{\mu}$

$\therefore x = b\text{cosh}(\sqrt{\mu}t) + \dfrac{V}{\sqrt{\mu}}\text{sinh}(\sqrt{\mu}t)$

When the particle reaches the origin O, we have $x = 0$ at $t = t_2$

So, $b\text{cosh}(\sqrt{\mu}t_2) + \dfrac{V}{\sqrt{\mu}}\text{sinh}(\sqrt{\mu}t_2) = 0$ or, $a\text{cos}(\sqrt{\mu}t_1)\text{cosh}(\sqrt{\mu}t_2) - a\text{sin}(\sqrt{\mu}t_1)\text{sinh}(\sqrt{\mu}t_2) = 0$.

[Using, $b = a\text{cos}(\sqrt{\mu}t_1)$ and $V = -a\sqrt{\mu}\text{sin}(\sqrt{\mu}t_1)$]

Hence, $\tan(\sqrt{\mu}t_1)\tanh(\sqrt{\mu}t_2) = 1$.

[Note that: For the first half, we can rewrite the equation (1) as $(D^2 - \mu)x = 0$.

The the general solution is $x = C_1\text{cos}(\sqrt{\mu}t) + D_1\text{sin}(\sqrt{\mu}t)$, where C_1 and D_1 are constants of integration.

$\dot{x} = \sqrt{\mu}\left(-C_1\text{sin}(\sqrt{\mu}t) + D_1\text{cos}(\sqrt{\mu}t)\right)$,

At A, $x = a$ and $\dot{x} = 0$ when $t = 0$, $\therefore C_1 = a$ and $D_1 = 0$.

$x = acos(\sqrt{\mu}t)$

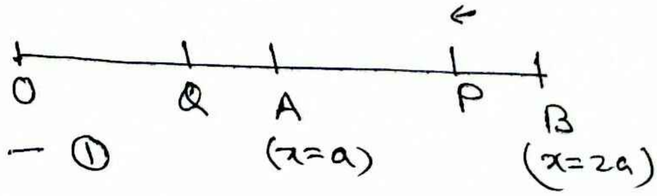
Thus at B, $b = a\text{cos}(\sqrt{\mu}t_1)$ and $V = -a\sqrt{\mu}\text{sin}(\sqrt{\mu}t_1)$.]

6. A particle of mass m moving in a straight line is acted on by an attractive force $m\mu a^2x^{-2}$ for $x \geq a$ and $m\mu xa^{-1}$ for $x < a$ towards a fixed point in the line, where x is the distance

of the particle measured from the fixed point and μ, a are positive constants. If the particle starts from rest at a distance $x = 2a$, then prove that it will reach the point $x = 0$ after a speed $\sqrt{2\mu a}$ after time $(1 + \frac{3\pi}{4})\sqrt{\frac{a}{\mu}}$.

Hints:

for motion from B to A,



$$mv\frac{dv}{dx} = -m\mu a^2 x^{-2} \quad — (1)$$

or, $v\,dv = -\mu a^2 x^{-2}\,dx$

Integrating, $\frac{v^2}{2} = \frac{\mu a^2}{x} + C_1$

At $x = 2a$, $v = 0$ $\therefore$ $C_1 = -\frac{\mu a}{2}$

Therefore, $\frac{v^2}{2} = \frac{\mu a^2}{x} - \frac{\mu a}{2}$ i.e $v^2 = \frac{\mu a}{x}(2a-x)$ — (2)

$\therefore$ $v = -\sqrt{\frac{\mu a}{x}(2a-x)}$ so, at $x = a$, $v^2 = \mu a$

$$\int_{2a}^{a}\sqrt{\frac{x}{2a-x}}\,dx = -\sqrt{\mu a}\int_0^{t_1} dt$$

or, $\int_0^{\pi/4} -4a\cos^2\theta\, d\theta = -\sqrt{\mu a}\,t_1$

Put $x = 2a\cos^2\theta$

$dx = -4a\cos\theta\sin\theta\, d\theta$

x	$2a$	a
θ	0	$\pi/4$

$\therefore$ $t_1 = 2\sqrt{\frac{a}{\mu}}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta$

$$= 2\sqrt{\frac{a}{\mu}}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = 2\sqrt{\frac{a}{\mu}}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \left(1+\frac{2\pi}{4}\right)\sqrt{\frac{a}{\mu}} \quad — (3)$$

for motion from A to O,

$$mv\frac{dv}{dx} = -m\mu a^{-1}x \quad — (4)$$

or, $v\,dv = -\mu a^{-1}x\,dx$

Integrating, $\frac{v^2}{2} = -\mu a^{-1}\frac{x^2}{2} + D_1$

At $x = a$, $v^2 = \mu a$ $\therefore$ $D_1 = \frac{\mu a}{2} + \frac{\mu a}{2} = \mu a$

Therefore, $\frac{v^2}{2} = -\frac{\mu a^{-1}}{2}x^2 + \mu a$ i.e. $v^2 = \frac{\mu}{a}(2a^2-x^2)$ — (5)

$\therefore$ $v = -\sqrt{\frac{\mu}{a}}\sqrt{2a^2-x^2}$ — (5a)

$$\int_a^0 \frac{dx}{\sqrt{2a^2-x^2}} = -\sqrt{\frac{\mu}{a}}\int_0^{t_2} dt \quad \text{or,} \quad \left(\sin^{-1}\frac{x}{\sqrt{2}a}\right)_a^0 = -\sqrt{\frac{\mu}{a}}\,t_2$$

$\therefore$ $t_2 = \sqrt{\frac{a}{\mu}}\,\frac{\pi}{4}$ — (6)

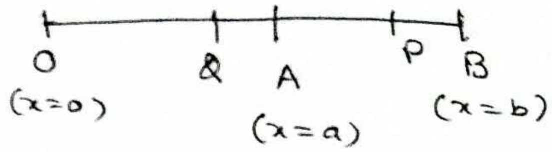
~~from (5a)~~, put $x = 0$ in (5a), we have $v = -\sqrt{2\mu a}$

from (3) and (6), we obtain, $t_1 + t_2 = \left(1 + 3\frac{\pi}{4}\right)\sqrt{\frac{a}{\mu}}$

7. A particle starts from rest at a distance $b(> a)$ from a fixed point O under the action of a force through the fixed point, the law of which at a distance x from O is $F = \mu\left(1 - \frac{a}{x}\right)$ toward O when $x > a$ but $F = \mu\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$ away from O when $x < a$.
Show that the velocity of the particle will again vanish when $x = \frac{a^2}{b}$.

Hints :

For motion from B to A,



$v\frac{dv}{dx} = -\mu\left(1 - \frac{a}{x}\right)$ — ①

$v\,dv = -\mu\left(1 - \frac{a}{x}\right)dx$

Integrating, $\frac{v^2}{2} = -\mu(x - a\log x) + C$

At $x = b$, $v = 0$ $\therefore C = \mu(b - a\log b)$

$\therefore v^2 = 2\mu(b - x) + 2\mu a\log\frac{x}{b}$ — ②

$\therefore$ At $x = a$, $v^2 = 2\mu(b-a) + 2\mu a\log\frac{a}{b}$ — ③

For motion from A to away from O,

$v\frac{dv}{dx} = \mu\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$

or, $v\,dv = \mu\left(\frac{a^2}{x^2} - \frac{a}{x}\right)dx$

Integrating, $\frac{v^2}{2} = \mu\left(-\frac{a^2}{x} - a\log x\right) + D$

At $x = a$, $v^2 = 2\mu(b-a) + 2\mu a\log\frac{a}{b}$

$\therefore D = \mu(b-a) + \mu a\log\frac{a}{b} + \mu a + \mu a\log a = \mu\left(b + a\log\frac{a^2}{b}\right)$

$\therefore v^2 = 2\mu\left(b - \frac{a^2}{x}\right) + \mu a\log\frac{a^2}{bx}$

$\therefore v = 0$ when $2\mu\left(b - \frac{a^2}{x}\right) = 0$ and $\log\frac{a^2}{bx} = 0$ i.e when $x = \frac{a^2}{b}$

8. A particle moves with a constant acceleration. Show that the space-average of the velocity over any distance is $\frac{2}{3}\frac{u_1^2 + u_1u_2 + u_2^2}{u_1 + u_2}$ and the time-average velocity is $\frac{1}{2}(u_1 + u_2)$, where u_1 and u_2 are the initial and final velocities.

Solution: We know that, $\overline{v}$(time average) $= \dfrac{\int_{t_1}^{t_2} vdt}{\int_{t_1}^{t_2} dt}$ and $\overline{v}$(space average) $= \dfrac{\int_{s_1}^{s_2} vds}{\int_{s_1}^{s_2} ds}$.

Let the particle moves with a constant acceleration f and with a initial velocity u_1, then its velocity at any instant t is given by $v = u_1 + ft$.

Also assume that T be the total time and so, the final velocity, $u_2 = u_1 + fT$

$$\therefore \overline{v}(\text{time average}) = \frac{\displaystyle\int_0^T (u_1 + ft)dt}{\displaystyle\int_0^T dt} = \frac{u_1T + 1/2fT^2}{T} = \frac{2u_1 + fT}{2} = \frac{u_1 + u_2}{2}$$

Let the particle travels s distance in time t under the constant acceleration f, then $v^2 = u_1^2 + 2fs$. Also assume that the particle travels S distance in T time and and so, $u_2^2 = u_1^2 + 2fS$.

$$\therefore \overline{v}(\text{space average}) = \frac{\displaystyle\int_0^S \sqrt{u_1^2 + 2fs}ds}{\displaystyle\int_0^S ds} = \frac{1}{\frac{3}{2}2f}\frac{[(u_1^2 + 2fS)^{3/2}]_0^S}{S} = \frac{2}{3}\frac{(u_1^2 + 2fS)^{3/2} - u_1^3}{2fS} =$$

$$\frac{2}{3}\frac{u_2^3 - u_1^3}{u_2^2 - u_1^2} = \frac{2}{3}\frac{u_1^2 + u_1u_2 + u_2^2}{u_1 + u_2}.$$

9. A particle moves with a constant velocity v_1 for t_1 seconds and with a constant velocity v_2 for t_2 seconds. Find the space-average of the velocity over any distance.

Hints: As the particle moves with constant velocity v_1 for t_1 seconds and and with a constant velocity v_2 for t_2 seconds,

$$\text{so } \overline{v}(\text{time average}) = \frac{\displaystyle\int_0^{t_1} v_1dt + \int_{t_1}^{t_1+t_2} v_2dt}{\displaystyle\int_0^{t_1+t_2} dt} = \frac{v_1t_1 + v_2(t_1 + t_2 - t_1)}{t_1 + t_2} = \frac{v_1t_1 + v_2t_2)}{t_1 + t_2}.$$

$$\text{Again, } \overline{v}(\text{space average}) = \frac{\displaystyle\int_0^{s_1} v_1ds + \int_{s_1}^{s_1+s_2} v_2ds}{\displaystyle\int_0^{s_1+s_2} ds} = \frac{v_1s_1 + v_2s_2}{s_1 + s_2} = \frac{v_1(v_1t_1) + v_2(v_2t_2)}{v_1t_1 + v_2t_2} =$$

$$\frac{v_1^2t_1 + v_2^2t_2}{v_1t_1 + v_2t_2}.$$

10. A particle is let fall from rest from a point outside the earth at a distance b from the centre. Prove that the square of the velocity of the particle on the reaching the centre is $ga\left(3 - \frac{2a}{b}\right)$, where a is the radius of the earth and g is the value of gravity at its surface.

[Hints: We know that the earth attracts a particle outside its surface with a force varying inversely as the square of the distance of the particle from its centre and also attracts a particle inside its surface with a force varying directly as its distance from its centre.

Given that a is the radius of the earth.

Outside the earth surface, the equation of motion of the particle is

$$v\frac{dv}{dx} = -\frac{\mu}{x^2} \tag{1}$$

Using the initial condition: at $x = b$, $v = 0$

$v^2 = 2\mu\left(\frac{1}{x} - \frac{1}{b}\right)$

On the surface of the earth, $\mu = ga^2$

So, at $x = a$, $v^2 = 2ga^2\left(\frac{1}{a} - \frac{1}{b}\right) = 2ga\left(1 - \frac{a}{b}\right)$

Inside the earth surface, the equation of motion of the particle is

$$v\frac{dv}{dx} = -\mu' x \tag{2}$$

On the surface of the earth, $g = \mu' a$

Using the condition: at $x = a$, $v^2 = 2ga\left(1 - \frac{a}{b}\right)$

$\therefore v^2 = -\frac{g}{a}x^2 + ga\left(3 - \frac{2a}{b}\right)$

So, at $x = 0$, $v^2 = ga\left(3 - \frac{2a}{b}\right)$.]

11. A particle is projected from the earth's surface vertically upwards with a velocity V. If h and H be the greatest heights attained by the particle moving under uniform and variable accelerations respectively, then show that $\frac{1}{h} - \frac{1}{H} = \frac{1}{R}$, where R is the radius of the earth.

Solution: We know that the earth attracts a particle outside its surface with a force varying inversely as the square of the distance of the particle from its centre and also attracts a particle inside its surface with a force varying directly as its distance from its centre.

Given that R is the radius of the earth.

If x be the distance of the particle from the earth's centre and v be its velocity at any time t, after it is projected from a point on the surface, then the acceleration of the particle is given by

$$v\frac{dv}{dx} = -\frac{\mu}{x^2} \tag{1}$$

where $\frac{\mu}{R^2} = g =$the acceleration due to gravity on the earth's surface

Writing this equation as $v\frac{dv}{dx} = -\frac{gR^2}{x^2}$.

Here, V is the velocity of projection of the particle.

Since, h is the greatest heights attained by the particle moving under uniform accelerations,

$$\int_V^0 vdv = \int_R^{h+R} -gdx \text{ or, } \int_0^V vdv = \int_R^{h+R} gdx \text{ or } \frac{V^2}{2} = gh \text{ i.e. } V^2 = 2gh \tag{2}$$

Since, H is the greatest heights attained by the particle moving under variable accelerations,

$$\int_V^0 vdv = \int_R^{H+R} -\frac{gR^2}{x^2}dx \text{ or, } \int_0^V vdv = \int_R^{H+R} \frac{gR^2}{x^2}dx \text{ or } \frac{V^2}{2} = [-\frac{gR^2}{x}]_R^{H+R}$$

$$\text{i.e. } V^2 = 2\left(-\frac{gR^2}{H+R} + \frac{gR^2}{R}\right) \text{ or, } V^2 = \frac{2gR^2H}{R(H+R)} = \frac{2gRH}{H+R} \tag{3}$$

From, equation (2) and (3), we can write that,

$$2gh = \frac{2gRH}{H+R} \text{ or } \frac{H+R}{RH} = \frac{1}{h} \text{ or, } \frac{1}{R} + \frac{1}{H} = \frac{1}{h} \therefore \frac{1}{h} - \frac{1}{H} = \frac{1}{R}.$$