Problems in particle dynamics

We know that the expression for the acceleration may be taken as $\ddot{x}(=\dfrac{d^2x}{dt^2})$ or $\dfrac{dv}{dt}$ or $v\dfrac{dv}{dx}$ where $v=\dfrac{dx}{dt}=\dot{x}$. A particle may move arbitrary in space. But here we basically consider the case where a particle moves along a straight line. To explore the motion of the particle, following things comes into mind:

1. Is there any initial velocity of the particle or not?
2. is the acceleration uniform or variable?
3. is the acceleration attractive or repulsive?

Motion in a straight line with constant acceleration

If a particle moves along a straight line starting from a fixed point on it with initial velocity u under the constant acceleration f, then

1. $v = u + ft$
2. $x = ut + \frac{1}{2}ft^2$
3. $v^2 = u^2 + 2fs$

[For proof, see any standard book.]
[Hints: For 1,
The equation of motion of the particle is
$\frac{dv}{dt} = f$, where f is constant
or, $dv = fdt$. Integrating, $v = ft + C$, where C is an arbitrary constant of integration.
Initially, at $t = 0$, $v = u$. So, $A = u$.
For 2 and 3, use $v = \frac{dx}{dt}$ and $f = v\frac{dv}{ds}$]

Motion in a straight line with constant acceleration (contd.)

Following conclusions for the vertical motion under gravity can be drawn
(i) when the particle falls freely from rest under gravity from a height h above the earth's surface

1. $v = gt$
2. $h = \frac{1}{2}gt^2$
3. $v^2 = 2gh$

(ii) when the particle is thrown vertically upwards with a velocity u and v be its velocity after time t at a height h above the starting point, then

1. $v = u - gt$
2. $x = ut - \frac{1}{2}gt^2$
3. $v^2 = u^2 - 2gs$

Question: What is the distance traveled by the particle in t^{th} second under uniform accelration f with the initial velocity u? [Ans: $u + (f(2t-1))/2$]

1. A particle moves in a straight line under the action of a repulsive force (i.e. away from a fixed point) proportional to the distance from the fixed point on that line. If the particle starts from rest at a distance a from the fixed point, then we can find that the particle will move along the straight line with increasing distance and velocity.
Note that the equation of motion of the particle is $\ddot{x} = \mu x$ where $\mu > 0$.
The general solution of it is $x = Acosh(\sqrt{\mu}t) + Bsinh(\sqrt{\mu}t)$, where A and B are the constants of integration.
Using the initial condition, $x = a$ and $\dot{x} = 0$ when $t = 0$.
$\therefore x = acosh(\sqrt{\mu}t)$ and $\dot{x} = a\sqrt{\mu}sinh(\sqrt{\mu}t)$.

2. A particle moves in a straight line under the action of a repulsive force (i.e. away from a fixed point) proportional to (the distance from the fixed point)^n on that line. Then the equation of motion of the particle will be $\ddot{x} = \mu x^n$ where $\mu > 0$.

Motion in a straight line under variable acceleration (contd.)

3. A particle moves in a straight line under the action of an attractive force (i.e. towards the fixed point) proportional to the distance from the fixed point on that line. If the particle starts from rest at a distance a from the fixed point, then we can find that the particle will move along the straight line with decreasing distance and velocity.
   Note that the equation of motion of the particle is $\ddot{x} = -\mu x$ where $\mu > 0$.
   The general solution of it is $x = Acos(\sqrt{\mu}t) + Bsin(\sqrt{\mu}t)$, where A and B are the constants of integration.
   Using the initial condition, $x = a$ and $\dot{x} = 0$ when $t = 0$.
   $\therefore x = acos(\sqrt{\mu}t)$ and $\dot{x} = -a\sqrt{\mu}sin(\sqrt{\mu}t)$

4. A particle moves in a straight line under the action of an attractive force (i.e. towards the fixed point) inversely proportional to (the distance from the fixed point)^n on that line. Then that the equation of motion of the particle is $\ddot{x} = -\dfrac{\mu}{x^n}$ where $\mu > 0$.

Question: A particle moving in a straight line is subjected to a resistance which produces the retardation kv^3, where v is the velocity and k is a constant. Show that the velocity v and the time t are given in terms of the distance s by the equations $v = \dfrac{u}{1+ksu}$, $t = \dfrac{s}{u} + \dfrac{1}{2}ks^2$, where u is the initial velocity.

Solution: Let s be the distance described by the particle in time t, and v be the velocity of the particle at that instant. Then, the equation of motion of the particle is

$$\frac{dv}{dt} = -kv^3 \tag{1}$$

for the retardation produced being kv^3

Equation (1) can be written as $\dfrac{-dv}{v^3} = kdt$.

Integrating, we have $\dfrac{1}{2v^2} = kt + C$, where C is an arbitrary constant of integration.

As u is the initial velocity, so we have $v = u$ when $t = 0$.

$\therefore C = 1/2u^2$.

So, we get

$$t = \frac{1}{2k}\left(\frac{1}{v^2} - \frac{1}{u^2}\right) \tag{2}$$

Again we have $\frac{dv}{dt} = v\frac{dv}{ds}$, so the equation (1) can also be written as

$$v\frac{dv}{ds} = -kv^3$$

$$\frac{-dv}{v^2} = -kds$$

Integrating, we get $\frac{1}{v} = ks + B$, where B is an arbitrary constant of integration.

As s is measured from the starting position of the particle, we have $s = 0$ when $v = u$. $\therefore B = 1/u$.

$$\therefore s = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{u}\right).$$

or, $ksu = \frac{u}{v} - 1$ or, $v = \frac{u}{1 + ksu}$. Putting the value of v in equation (2), we get, $t = \frac{1}{2k}\left\{\frac{(1+ksu)^2 - 1}{u^2}\right\} = \frac{1}{2}ks^2 + \frac{s}{u}$. Hence the result.

1. A particle starts from rest at a distance $b(> a)$ from a fixed point O under the action of a force through the fixed point, the law of which at a distance x from O is $F = \mu\left(1 - \frac{a}{x}\right)$ toward O when $x > a$ but $F = \mu\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$ away from O when $x < a$.

   Show that the velocity of the particle will again vanish when $x = \frac{a^2}{b}$.
2. A particle moves in a straight line under a force towards a fixed point O on the straight line. The magnitude of the force is equal to $\frac{\mu}{x^2} - \frac{\lambda}{x^3}$ when the particle is at a distance x from O. Show that it oscillates between this distance and the distance $\frac{\lambda a}{2\mu a - \lambda}$ and that its periodic time is $\frac{2\pi\mu a^3}{(2\mu a - \lambda)^{\frac{3}{2}}}$.
3. A particle moves along X-axis with an acceleration $\frac{\mu}{x^3}$, where $\mu(> 0)$ is constant and x is the distance from origin, If it starts from rest at $x = a$ and the acceleration is towards the origin. Find the time taken to reach $x = a/4$ from $x = 3a/4$.

4. A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Show that the time it takes in reaching a height h is $\frac{1}{3}\sqrt{\frac{2a}{g}}\left\{\left(1+\frac{h}{a}\right)^{\frac{3}{2}}-1\right\}$.

5. A particle of mass m moving in a straight line is acted on by an attractive force $m\mu a^2x^{-2}$ for $x \geq a$ and $m\mu xa^{-1}$ for $x < a$ towards a fixed point in the line, where x is the distance of the particle measured from the fixed point and μ, a are positive constants. If the particle starts from rest at a distance $x = 2a$, then prove that it will reach the point $x = 0$ after a speed $\sqrt{2\mu a}$ after time $(1+\frac{3\pi}{4})\sqrt{\frac{a}{\mu}}$.

6. A particle moves along the axis of X starting from rest at $x = a$; for an interval t_1 from the beginning of the motion the acceleration is $(-\mu x)$, for a subsequent time t_2 the acceleration is μx and at the end of this interval the particle is at the origin again. Prove that $tan(\sqrt{\mu}t_1)tanh(\sqrt{\mu}t_2) = 1$.

Questions (contd.)

7. A particle moves with a constant acceleration. Show that the space-average of the velocity over any distance is $\frac{2}{3}\frac{u_1^2 + u_1 u_2 + u_2^{-2}}{u_1 + u_2}$ and the time-average velocity is $\frac{1}{2}(u_1 + u_2)$, where u_1 and u_2 are the initial and final velocities.
8. A particle is let fall from rest from a point outside the earth at a distance b from the centre. Prove that the square of the velocity of the particle on the reaching the centre is $ga\left(3 - \frac{2a}{b}\right)$, where a is the radius of the earth and g is the value of gravity at its surface.

References

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