Sequence of Real Numbers

Questions:

Q1. Find the limit of these sequences:

(a) $\{\frac{1}{n+1}\}$, (b) $\{\frac{3n}{n+1}\}$, (c) $\{\frac{n+1}{2n+1}\}$, (d) $\{\frac{5n+3}{4n+1}\}$, (e) $\{(1+\frac{1}{n})^n\}$, (f) $\{\frac{1}{n}\sin\frac{n\pi}{2}\}$, (g) $\{n^2\cos\frac{n\pi}{2}\}$, (h) $\{\frac{1}{2^n}\}$, (i) $\{\sqrt{n}[\sqrt{n+1}-\sqrt{n}]\}$, (j) $\{\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+...+\frac{n}{n^2}\}$, (k) $\{\frac{5n^2+4}{3n^2+2}\}$, (l) $\{1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{n-1}}\}$

Solution: Let $\epsilon > 0$

Let $f(n) = \frac{1}{n}$

Now, $|f(n) - 0| = \frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$.

Let, m=$[\frac{1}{\epsilon} + 1]$.

Then, m is a positive integer and $|f(n) - 0| < \epsilon$ for all $n \geq m$. Hence, we can say that $\{f(n)\}$ is convergent and $\lim_{n\to\infty} f(n) = 0$. That is, $\lim_{n\to\infty} \frac{1}{n} = 0$.

Debprasad Pal, Bethune College

(a) We know that, $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n} = \frac{0}{1+0} = 0$.

(b) Try yourself [Ans: 3]

(c) Try yourself [Ans: $\frac{1}{2}$]

(d) Try yourself [Ans: $\frac{5}{4}$]

(e) Let $f(n) = (1+\frac{1}{n})^n$

Then, $f(n) = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + ... + \frac{n(n-1)(n-2)...1}{n!}(\frac{1}{n})^n$

$= 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + ... + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})...(1-\frac{n-1}{n})$

So, $f(n+1) = 1+1+\frac{1}{2!}(1-\frac{1}{n+1})+\frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})+....+\frac{1}{n!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})(1-\frac{3}{n+1})...(1-\frac{n+1-2}{n+1}) + \frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})(1-\frac{3}{n+1})...(1-\frac{n+1-1}{n+1})$

Note that the first two terms of f(n) and f(n+1) are same.

Also, $(1-\frac{1}{n+1}) > (1-\frac{1}{n})$, for all $n \in \mathbb{N}$.

Therefore, $f(n+1) - f(n) > 0$ for all $n \geq 3$.

Thus, {f(n)} is a strictly monotone increasing sequence.

Now, since $(1-\frac{1}{n}) < 1$, for all $n \in \mathbb{N}$,

$f(n) < 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + ... + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})...(1-\frac{n-1}{n}) < 1 + 1 + \frac{1}{2!} + +\frac{1}{3!} + ... + \frac{1}{n!}$

As, $3! > 2^2,\ 4! > 2^3,\ ....,n! > 2^{(n-1)}$ for all $n \geq 3$

so, $f(n) < 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}} = 1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 3-\left(\frac{1}{2}\right)^{n-1}$

i.e. $f(n) < 3$ for all $n \geq 3$

Also, $f(1) = 2$. Since {f(n)} is strictly monotone increasing, so, $f(n) \geq 2$ for all $n \in \mathbb{N}$

Therefore, $2 \leq f(n) < 3$, for all $n \in \mathbb{N}$.

Hence, the sequence is bounded.

Since, a bounded monotone sequence is convergent, thus {f(n)} is convergent.

The limit of the sequence is denoted by e. Therefore, $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

(f) Since, $-1 \leq \sin\frac{n\pi}{2} \leq 1$ for all $n \in \mathbb{N}$, therefore $-\frac{1}{n} \leq \frac{1}{n}\sin\frac{n\pi}{2} \leq \frac{1}{n}$, for all $n \in \mathbb{N}$.

Now, since $\lim_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty}\frac{-1}{n} = 0$

Hence, by Sandwich theorem, $\lim_{n\to\infty}\frac{1}{n}\sin\frac{n\pi}{2} = 0$.

(g) Since $\cos\frac{n\pi}{2}$ oscillates between -1 and 1 and $\lim_{n\to\infty} n^2 = \infty$, thus $\{n^2\cos\frac{n\pi}{2}\}$ oscillates infinitely.

(h) Let $\epsilon > 0$

Let $f(n) = \frac{1}{2^n}$

Now, $|f(n) - 0| = \frac{1}{2^n} < \epsilon$ if $2^n > \frac{1}{\epsilon}$ or $nlog2 > log(1/\epsilon)$ i.e. if $n > \frac{log(1/\epsilon)}{log2}$.

Debprasad Pal, Bethune College

Let, m=$[\frac{log(1/\epsilon)}{log2}+1]$.

Then, m is a positive integer and $|f(n) - 0| < \epsilon$ for all $n \geq m$. Hence, we can say that $\{f(n)\}$ is convergent and $\lim_{n\to\infty} f(n) = 0$. That is, $\lim_{n\to\infty}\frac{1}{2^n} = 0$.

(i) Let, $f(n) = \sqrt{n}(\sqrt{n+1}-\sqrt{n})$

Then, $f(n) = \frac{\sqrt{n}(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}-\sqrt{n})} = \frac{\sqrt{n}(n+1-n)}{(\sqrt{n+1}+\sqrt{n})} = \frac{\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})}$,

$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\frac{\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})} = \lim_{n\to\infty}\frac{1}{(\sqrt{1+1/n}+1)} = \frac{1}{(\sqrt{1+\lim_{n\to\infty}1/n}+1)} = \frac{1}{(\sqrt{1+0}+1)} = \frac{1}{2}$ [Since $\lim_{n\to\infty}\frac{1}{n} = 0$]

(j) Let, $f(n) = \frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+...+\frac{n}{n^2} = \frac{1+2+3+...+n}{n^2} = \frac{\frac{n(n+1)}{2}}{n^2} = \frac{(1+1/n)}{2}$

Since, $1+2+3+...+n = \frac{n(n+1)}{2}$

We know that $\lim_{n\to\infty}\frac{1}{n} = 0$

Therefore $\lim_{n\to\infty} f(n) = 1/2$

(k) Let, $f(n) = \dfrac{5n^2+4}{3n^2+2}$.

Now, $f(n) = \dfrac{5n^2+4}{3n^2+2} = \dfrac{5+4.\frac{1}{n}.\frac{1}{n}}{3+2.\frac{1}{n}.\frac{1}{n}} = \frac{g(n)}{h(n)}$, where $g(n) = 5+4.\frac{1}{n}.\frac{1}{n}$ and $h(n) = 3+2.\frac{1}{n}.\frac{1}{n}$

$\lim_{n\to\infty} g(n) = \lim_{n\to\infty}(5+4.\frac{1}{n}.\frac{1}{n}) = 5$ and $\lim_{n\to\infty} h(n) = \lim_{n\to\infty}(3+2.\frac{1}{n}.\frac{1}{n}) = 3$, since $\lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \dfrac{g(n)}{h(n)} = \dfrac{\lim_{n\to\infty} g(n)}{\lim_{n\to\infty} h(n)} = \dfrac{5}{3}$.

(l) Try yourself [Ans. 2]

Note that: All the above sequences are convergent. A convergent sequence is Cauchy and vice-versa.

Q2. Using the Cauchy's general principle of convergence to show that the sequence (a) $\{\frac{n}{n+1}\}$ is convergent but (b) $\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\}$ is not.

Solution:

Cauchys general principle of convergence: A necessary and sufficient condition for the convergence of a sequence $\{f(n)\}$ is that for any given $\epsilon > 0$, there exists a positive integer m such that $|f(n+p) - f(n)| < \epsilon$, for all $n \geq m$ and for $p \in \mathbb{N}$.

Debprasad Pal, Bethune College

(a) Let $\epsilon > 0$

Let $f(n) = \dfrac{n}{n+1}$

Now, for $p \in \mathbb{N}$, $|f(n+p) - f(n)| = |\dfrac{n+p}{(n+p)+1} - \dfrac{n}{n+1}| = |\dfrac{(n+p)(n+1) - n(n+p+1)}{(n+p+1)(n+1)}| = |\dfrac{p}{(n+p+1)(n+1)}| < \dfrac{1}{n+1}$, since $\dfrac{p}{n+p+1} < 1$ for every $p, n \in \mathbb{N}$

So, $|f(n+p) - f(n)| < \dfrac{1}{n+1} < \dfrac{1}{n} < \epsilon$ if $n > \dfrac{1}{\epsilon}$.

Let, m=$[\frac{1}{\epsilon}+1]$.

Then, m is a positive integer and $|f(n+p) - f(n)| < \epsilon$ for all $n \geq m$ and p $\in \mathbb{N}$.

Using the Cauchy's general principle of convergence, we can say that {f(n)} is convergent.

(b) Let, $\epsilon = 0.1$

Let $f(n) = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$

Now, $f(n+p) - f(n) = \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+\frac{1}{(n+1)}+...+\frac{1}{(n+p)}\right) - \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right) = \frac{1}{(n+1)}+\frac{1}{(n+2)}+...+\frac{1}{(n+p)}$

Let us choose n=k and p=k

Then, $|f(2k) - f(k)| = \frac{1}{(k+1)}+\frac{1}{(k+2)}+...+\frac{1}{(2k)} > \frac{1}{(2k)}+\frac{1}{(2k)}+...+\frac{1}{(2k)}$(k times) $= \frac{1}{2}$

Hence for the chosen $\epsilon = 0.1$, no positive integer m can be found for which $|f(n+p) - f(p)| < \epsilon$, for all $n \geq m$ and p $\in \mathbb{N}$. This shows that the Cauchy's general principle of convergence is not satisfied by the sequence and therefore, we can say that {f(n)} is not convergent.

Q3. Show that the sequence $\{\frac{3n+1}{n+2}\}$ is monotone increasing and bounded above. Also show that it is convergent and find its limit.

Solution: Let $u_n = \frac{3n+1}{n+2}$

Now, $u_{n+1} - u_n = \frac{3(n+1)+1}{(n+1)+2} - \frac{3n+1}{n+2} = \frac{(3n+4)(n+2) - (3n+1)(n+3)}{(n+3)(n+2)}$

$= \frac{5}{(n+3)(n+2)} > 0$ for all $n \in \mathbb{N}$

So, the sequence is strictly monotone increasing.

Now, $u_n - 3 = \frac{3n+1}{n+2} - 3 = \frac{-5}{n+2} < 0$ for all $n \in \mathbb{N}$

Therefore, it is bounded above. We know that if a sequence is monotone increasing and bounded above, then it is convergent. As a result, $\{u_n\}$ is convergent as it is both monotone increasing and bounded above.

Let $\epsilon > 0$

$|u_n - 3| = |\frac{3n+1}{n+2} - 3| = \frac{5}{n+2} < \frac{5}{n} < \epsilon$ if $n > \frac{5}{\epsilon}$ i.e. $n > \frac{5}{\epsilon}$.

Let m=$[\frac{5}{\epsilon}+1]$.

Then m is a positive integer.

Debprasad Pal, Bethune College

Thus, for any given $\epsilon > 0$, there exists a positive integer m such that $|u_n - 3| < \epsilon$, for all n$\geq$m. Thus, $\lim_{n\to\infty} u_n = 3$. Hence, the sequence $\{u_n\}$ converges to 3.

[Note that, $u_n - 3$ is considered as 3 is the limit of the sequence. The limit of the sequence can also be founded as

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3n+1}{n+2} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{2}{n}} = \frac{3+0}{1+0} = 3, \text{ since } \lim_{n\to\infty} \frac{1}{n} = 0.]$$

Q4. Show that the sequence $\{x_n\}_n$, where $x_n = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)}$ is monotone increasing and bounded above. Also show that it is convergent and find its limit.

Solution: Here, $x_n = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)}$

Then, $x_{n+1} - x_n = \left(\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)}\right) -$

$\left(\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}\right)=\frac{1}{(2n+1)(2n+3)}>0$, for all $n\in\mathbb{N}$

Thus, $x_{n+1}>x_n$, for all $n\in\mathbb{N}$ and so $\{x_n\}$ is strictly monotone increasing.

Now, $x_n=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{3-1}{1.3}+\frac{5-3}{3.5}+\frac{7-5}{5.7}+...+\frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)}\right)=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+....-\frac{1}{2n+1}\right)=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{2n+1-1}{4n+2}=\frac{n}{2n+1}$

So, $x_n-\frac{1}{2}=\frac{n}{2n+1}-\frac{1}{2}=\frac{2n-(2n+1)}{2(2n+1)}=-\frac{1}{2(2n+1)}<0$ i.e. $x_n<1/2$, for all $n\in\mathbb{N}$ and consequently $\{x_n\}$ is bounded above.

Let $\epsilon>0$

Now, $|x_n-\frac{1}{2}|=|\frac{n}{2n+1}-\frac{1}{2}|=|\frac{2n-(2n+1)}{2(2n+1)}|=\frac{1}{2(2n+1)}<\frac{1}{4n}<\epsilon$ if $n>\frac{1}{4\epsilon}$.

Let, m=$[\frac{1}{4\epsilon}+1]$.

Then, m is a positive integer and $|x_n-\frac{1}{2}|<\epsilon$ for all $n\geq m$. Hence, we can say that $\{x_n\}$ is convergent and it converges to $\frac{1}{2}$.

[Note that we can show that x_n is monotone increasing in a different way. As, $x_n=\frac{n}{2n+1}$, so $x_{n+1}-x_n=\frac{n+1}{2(n+1)+1}-\frac{n}{2n+1}=\frac{(n+1)(2n+1)-n(2n+3)}{(2n+1)(2n+3)}=\frac{1}{(2n+1)(2n+3)}>0$, for all $n\in\mathbb{N}$ i.e. $x_{n+1}>x_n$, for all $n\in\mathbb{N}$.

$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2+0}=\frac{1}{2}$, since $\lim_{n\to\infty}\frac{1}{n}=0$.]]

Debaprasad Pal, Bethune College

Q5. Show that the sequence $\{\frac{5n+3}{4n+1}\}$ is strictly monotone decreasing and bounded below. Is the sequence convergent? Justify your answer.

Solution: Let $u_n=\frac{5n+3}{4n+1}$

Now, $u_{n+1}-u_n=\frac{5(n+1)+3}{4(n+1)+1}-\frac{5n+3}{4n+1}=\frac{(5n+8)(4n+1)-(5n+3)(4n+5)}{(4n+5)(4n+1)}=\frac{-7}{(4n+5)(4n+1)}<0$, for all $n\in\mathbb{N}$

Thus, $u_{n+1}<u_n$, for all $n\in\mathbb{N}$ and so $\{u_n\}$ is strictly monotone decreasing.

Also, $u_n-\frac{5}{4}=\frac{5n+3}{4n+1}-\frac{5}{4}=\frac{7}{4(4n+1)}>0$, for all $n\in\mathbb{N}$

i.e. $u_n>\frac{5}{4}$, for all $n\in\mathbb{N}$ and hence $\{u_n\}$ is bounded below. We know that a monotone decreasing sequence which is bounded below is convergent. As a result, $\{u_n\}$ is convergent. $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{5n+3}{4n+1}=\lim_{n\to\infty}\frac{5+\frac{3}{n}}{4+\frac{1}{n}}=\frac{5+0}{4+0}=\frac{5}{4}$, since $\lim_{n\to\infty}\frac{1}{n}=0$.]

Q6. Show that $\{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}\}$ is convergent

Solution:

Let $f(n) = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}$

Now, $f(n+1) - f(n) = \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) = \frac{1}{(n+1)!} > 0$, for all $n \in \mathbb{N}$

Thus, $f(n+1) > f(n)$, for all $n \in \mathbb{N}$ and so $\{f(n)\}$ is strictly monotone increasing.

As, $3! > 2^2,\ 4! > 2^3,\ ....,n! > 2^{(n-1)}$ for all $n \geq 3$

so, $f(n) < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n-1}} = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{n-1}$ i.e. $f(n) < 2$ for all $n \in \mathbb{N}$

Also, $f(1) = 1$. Since {f(n)} is strictly monotone increasing, so, $f(n) \geq 1$ for all $n \in \mathbb{N}$

Therefore, $1 \leq f(n) < 2$, for all $n \in \mathbb{N}$

Hence, the sequence is bounded.

Since, it is also monotone increasing, thus {f(n)} is convergent.

Debprasad Pal, Bethune College