Sequence: A sequence is a mapping or a function from $\mathbb{N}$ to $\mathbb{R}$. So, the mapping f:$\mathbb{N}\rightarrow\mathbb{R}$ is a real sequence and it is denoted by $\{f(n)\}$ or $\{f(1), f(2), f(3), ..., f(n), ...\}$. For example, the sequences u:$\mathbb{N}\rightarrow\mathbb{R}$ and x:$\mathbb{N}\rightarrow\mathbb{R}$ are denoted by $\{u_n\}_n$ (or $\{u_n\}$) and $\{x_n\}_n$ (or $\{x_n\}$) respectively.

Examples: Some example of sequences $\{x_n\}, n \in \mathbb{N}$ are given below, where x_n equals to any one of the following

1. c
2. n
3. $\frac{1}{n}$
4. $(-1)^n$
5. $(1+\frac{1}{n})^n$
6. $\frac{5n+2}{4n+3}$
7. $sin\frac{n\pi}{2}$
8. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+..+\frac{1}{n}$

Range: Let {f(n)} be a sequence of real numbers.
Then the range of the sequence is defined as $\{f(n) : n \in \mathbb{N}\}$

Bounds of a sequence: A sequence $\{f(n)\}$ is said to bounded above if there exists a real number B such that $f(n) \leq B$, for all $n \in \mathbb{N}$. It is said to bounded below if there exists a real number b such that $f(n) \geq b$, $\forall\ n \in \mathbb{N}$. If the sequence is both bounded below and bounded above, then it is said to be bounded i.e. there exists real numbers b and B for which $b \leq f(n) \leq B$, $\forall\ n \in \mathbb{N}$.

Monotone sequence: A sequence $\{f(n)\}$ is said to be monotone increasing if $f(n+1) \geq f(n)$, $\forall n \in \mathbb{N}$. If $f(n+1) > f(n)$, $\forall n \in \mathbb{N}$, then it is said to be strictly monotone increasing. Sequence $\{f(n)\}$ is said to be monotone decreasing if $f(n+1) \leq f(n)$, $\forall n \in \mathbb{N}$. For strictly monotone decreasing sequence, $f(n+1) < f(n)$ holds $\forall n \in \mathbb{N}$. A sequence is said to monotone if it is either monotone increasing or decreasing.

Limit of a sequence: A sequence $\{f(n)\}$ is said to have a limit l if for any given $\epsilon > 0$, there exists a positive integer m (depending upon ϵ) such that $|f(n) - l| < \epsilon$, $\forall n \geq m$ and we write it as $\lim_{n\to\infty} f(n) = l$. If l is finite, then we say that the sequence is convergent and it converges to l.

A sequence $\{f(n)\}$ is said to be a null sequence if $\lim_{n\to\infty} f(n) = 0$.

Question. Show that $\{\frac{1}{n}\}$ is a null sequence or $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution: Let $\epsilon > 0$

Let $f(n) = \frac{1}{n}$

Now, $|f(n) - 0| = \frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$.

Let, m=$[\frac{1}{\epsilon} + 1]$.

Then, m is a positive integer and $|f(n) - 0| < \epsilon$ for all $n \geq m$. Hence, we can say that $\{f(n)\}$ is convergent and $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, $\{\frac{1}{n}\}$ is a null sequence.

Limit theorems: Let $\{f(n)\}$ and $\{g(n)\}$ be two convergent sequences that converges to l and m respectively. Then,

(i) $\lim_{n\to\infty}(f(n)+g(n)) = l+m$;

(ii) if $c \in \mathbb{R}$, $\lim_{n\to\infty} cf(n) = cl$;

(iii) $\lim_{n\to\infty} f(n)g(n) = lm$;

(iv) if $g(n) \neq 0$, for all $n \in \mathbb{N}$ and if $m \neq 0$, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{l}{m}$;

Sandwich theorem: Let $\{f(n)\}, \{g(n)\}$ and $\{h(n)\}$ be three sequences of real numbers and there is a positive integer m such that $f(n) < g(n) < h(n)$, for all $n \geq m$. If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} h(n) = l$, then $\{g(n)\}$ is convergent and $\lim_{n\to\infty} g(n) = l$.

Theorem: The limit of a convergent sequence is unique.

Proof: If possible, let l_1 and l_2 be two distinct limits of the sequence $\{f(n)\}$.
Without any loss of generality, let $l_2 > l_1$.
Let us choose $\epsilon = \dfrac{l_2 - l_1}{2}$
Now, since l_1 is a limit of the sequence, so for $\epsilon > 0$ there exists a positive integer m_1 such that $|f(n) - l_1| < \epsilon$, for all $n \geq m_1$. (1)
Again, since l_2 is a limit of the sequence, so for $\epsilon > 0$ there exists a positive integer m_2 such that $|f(n) - l_2| < \epsilon$, for all $n \geq m_2$. (2)
Let m=max$\{m_1, m_2\}$
Then, for all $n \geq m$, $|f(n) - l_1| < \epsilon$ and $|f(n) - l_2| < \epsilon$.
Now, for all $n \geq m$,
$|l_2 - l_1| = |l_2 - f(n) + f(n) - l_1| = |\,(f(n) - l_1) - (f(n) - l_2)\,| \leq$
$|f(n) - l_1| + |f(n) - l_2| < 2\epsilon$ i.e. $\dfrac{l_2 - l_1}{2} < \epsilon$ which is a contraction.
Thus, our assumption is wrong.
Hence, the limit of a convergent sequence is unique.

Theorem: A convergent sequence is bounded.

Proof: Let {f(n)} be a convergent sequence and $\lim_{n\to\infty} f(n) = l$.
Let us choose $\epsilon = 1$.
Then there exists a positive integer m such that $|f(n) - l| < 1$, for all $n \geq m$.
Thus, $l - 1 < f(n) < l + 1$, for all $n \geq m$.
Now, let b=min$\{l - 1$, f(1), f(2),f(3),....,f(m-1)}, and
B=max$\{l + 1$, f(1), f(2),f(3),....,f(m-1)}.
Therefore, $b \leq f(n) \leq B$, for all $n \in \mathbb{N}$. Hence, the sequence {f(n)} is bounded.

Note: The converse of the above result is not true i.e. a bounded sequence may not be convergent. For example, let $f(n) = (-1)^n$, for all $n \in \mathbb{N}$. So, $-1 \leq f(n) \leq 1$, for all $n \in \mathbb{N}$. Hence the sequence is bounded. But the sequence is not convergent.

Theorem: A monotone increasing sequence which is bounded above is convergent and it converges to the least upper bound (supremum).

Proof: Let M be the supremum of the monotone increasing sequence $\{f(n)\}$ and it is bounded above.
Since M is the supremum of $\{f(n)\}$, so (i) $f(n) \leq M$, for all $n \in \mathbb{N}$ and (ii) for any given $\epsilon > 0$, there exists a positive integer m such that $f(m) > M - \epsilon$.
Therefore, $M - \epsilon < f(m) \leq f(m+1) \leq f(m+2) \leq f(m+3) \leq ...$
i.e. $M - \epsilon < f(n)$, for all $n \geq m$.
Since, $f(n) \leq M$, for all $n \in \mathbb{N}$ and so, $f(n) \leq M + \epsilon$, for all $n \in \mathbb{N}$.
Therefore $M - \epsilon \leq f(n) \leq M + \epsilon$, for all $n \geq m$.
Hence, the sequence $\{f(n)\}$ is convergent and it converges to the supremum M.

Theorem: A monotone decreasing sequence which is bounded below is convergent and it converges to the greatest lower bound (infimum).

Proof: Let l be the infimum of the monotone decreasing sequence {f(n)} and it is bounded below.
Since l is the infimum of {f(n)}, so (i) $l \leq f(n)$, for all $n \in \mathbb{N}$ and (ii) for any given $\epsilon > 0$, there exists a positive integer m such that $f(m) < l + \epsilon$.
Therefore, $....f(m+3) \leq f(m+2) \leq f(m+1) \leq f(m) < l + \epsilon$
i.e. $f(n) < l + \epsilon$, for all $n \geq m$.
Since, $l \leq f(n)$, for all $n \in \mathbb{N}$ and so, $l - \epsilon < f(n)$, for all $n \in \mathbb{N}$.
Therefore $l - \epsilon \leq f(n) \leq l + \epsilon$, for all $n \geq m$.
Hence, the sequence {f(n)} is convergent and it converges to the infimum l.

Note: Combining these last two theorems, we can say that a bounded monotone sequence is convergent.

Oscillatory and divergent sequences A sequence {f(n)} of real numbers is said to diverge to ∞ if for any given positive number G, there exists a positive integer m such that $f(n) > G$, for all $n \geq m$. We write it as $\lim_{n\to\infty} f(n) = \infty$ and we also say that the sequence {f(n)} tends to ∞.
A sequence {f(n)} of real numbers is said to diverge to $-\infty$ if for any given positive number G, there exists a positive integer m such that $f(n) < -G$, for all $n \geq m$. We write it as $\lim_{n\to\infty} f(n) = -\infty$ and we also say that the sequence {f(n)} tends to $-\infty$.
A bounded sequence which is not convergent is said to be a oscillatory sequence of finite oscillation. An unbounded sequence which is not divergent is said to be a oscillatory sequence of infinite oscillation. An oscillatory sequence is neither convergent nor divergent.

Examples:

1. $\{2^n\}$ diverges to ∞.
2. $\{-n^2\}$ diverges to $-\infty$.
3. $\{(-1)^n\}$ is an oscillatory sequence of finite oscillation.
4. $\{(-1)^n n\}$ is an oscillatory sequence of infinite oscillation.

Theorem: An unbounded monotone increasing sequence diverges to $+\infty$.

Theorem: An unbounded monotone decreasing sequence diverges to $-\infty$.

Theorem (Cauchy's general principle of convergence): A necessary and sufficient condition for the convergence of a sequence $\{f(n)\}$ is that for any given $\epsilon > 0$, there exists a positive integer m such that $|f(n+p) - f(n)| < \epsilon$, for all $n \geq m$ and for $p \in \mathbb{N}$.

Cauchy Sequence: A sequence $\{f(n)\}$ is said to be Cauchy sequence if for any given $\epsilon > 0$, there exists a positive integer m such that $|f(n+p) - f(n)| < \epsilon$, for all $n \geq m$ and for $p \in \mathbb{N}$.

Question: Show that a constant sequence is a Cauchy sequence.

Solution: Let c $\in \mathbb{R}$ and $f(n) = c$ for all $n \in \mathbb{N}$

Let us choose $\epsilon > 0$

Now for any $n \in \mathbb{N}$, $|f(n+p) - f(n)| = |c - c| = 0 < \epsilon$, where $p \in \mathbb{N}$.

This show that {f(n)} is a Cauchy sequence.

Theorem: A Cauchy sequence of real numbers is convergent and conversely.

Sequence	Bounded		Monotone		Convergent/Cauchy	
	Yes	No	Increasing	Decreasing	Yes [**Limit**]	No
$\{\frac{1}{n}\}$	✓	–	–	✓	Yes [**0**]✓	–
$\{(1+\frac{1}{n})^n\}$	✓	–	✓	–	Yes [**e**] ✓	–
$\{\frac{5n+3}{4n+1}\}$	✓	–	–	✓	Yes [$\frac{5}{4}$] ✓	–
$\{\frac{3n+1}{n+2}\}$	✓	–	✓	–	Yes [**3**] ✓	–
$\{\frac{n}{n+1}\}$	✓	–	✓	–	Yes [**1**] ✓	–
$\{(-1)^n\}$	✓	–	–	–	–	✓
$\{(-1)^n n\}$	–	✓	–	–	–	✓
$\{n^2\}$	–	✓	✓	–	–	✓
$\{-n\}$	–	✓	–	✓	–	✓

Table: In this table, we briefly describe the nature of some sequences.

Some results:

(i) $\lim_{n\to\infty} 1/n = 0$;
(ii) $\lim_{n\to\infty} n^{1/n} = 1$;
(iii) $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$.

References

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«Questions?»