Electrostatics

1. The Electrostatic Field

To calculate the force exerted by some electric charges, q_1, q_2, q_3, ... (**the source charges**) on another charge Q (**the test charge**) we can use the **principle of superposition**. This principle states that the interaction between any two charges is completely unaffected by the presence of other charges. The force exerted on Q by q_1, q_2, and q_3 (see Figure 1) is therefore equal to the vector sum of the force $\overline{F}_1$ exerted by q_1 on Q, the force $\overline{F}_2$ exerted by q_2 on Q, and the force $\overline{F}_3$ exerted by q_3 on Q.

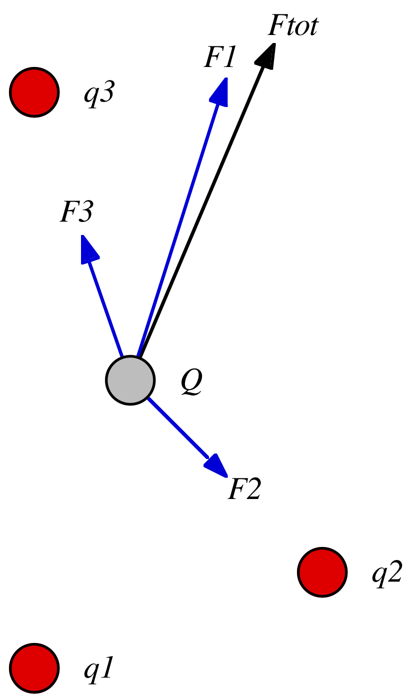

Figure 1. Superposition of forces.

The force exerted by a charged particle on another charged particle depends on their separation distance, on their velocities and on their accelerations. In this Chapter we will consider the special case in which the source charges are stationary.

The **electric field** produced by stationary source charges is called and **electrostatic field**. The electric field at a particular point is a vector whose magnitude is proportional to the total force acting on a test charge located at that point, and whose direction is equal to the direction of the force acting on a positive test charge. The electric field $\overline{E}$, generated by a collection of source charges, is defined as

$$\overline{E} = \frac{\overline{F}}{Q}$$

where $\overline{F}$ is the total electric force exerted by the source charges on the test charge Q. It is assumed that the test charge Q is small and therefore does not change the distribution of the source charges. The total force exerted by the source charges on the test charge is equal to

$$\overline{F} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \ldots = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \frac{q_3 Q}{r_3^2}\hat{r}_3 + \ldots\right) = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

The electric field generated by the source charges is thus equal to

$$\overline{E} = \frac{\overline{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

In most applications the source charges are not discrete, but are distributed continuously over some region. The following three different distributions will be used in this course:

1. **line charge** λ: the charge per unit length.

2. **surface charge** σ: the charge per unit area.

3. **volume charge** ρ: the charge per unit volume.

To calculate the electric field at a point $\overline{P}$ generated by these charge distributions we have to replace the summation over the discrete charges with an integration over the continuous charge distribution:

1. for a line charge: $\overline{E}(\overline{P}) = \frac{1}{4\pi\varepsilon_0}\int_{Line}\frac{\hat{r}}{r^2}\lambda dl$

2. for a surface charge: $\overline{E}(\overline{P}) = \frac{1}{4\pi\varepsilon_0}\int_{Surface}\frac{\hat{r}}{r^2}\sigma da$

3. for a volume charge: $\overline{E}(\overline{P}) = \frac{1}{4\pi\varepsilon_0}\int_{Volume}\frac{\hat{r}}{r^2}\rho d\tau$

Here $\hat{r}$ is the unit vector from a segment of the charge distribution to the point $\overline{P}$ at which we are evaluating the electric field, and r is the distance between this segment and point $\overline{P}$.

Example: Problem 2

a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges q a distance d apart. Check that your result is consistent with what you would expect when $z \gg d$.

b) Repeat part a), only this time make he right-hand charge $-q$ instead of $+q$.

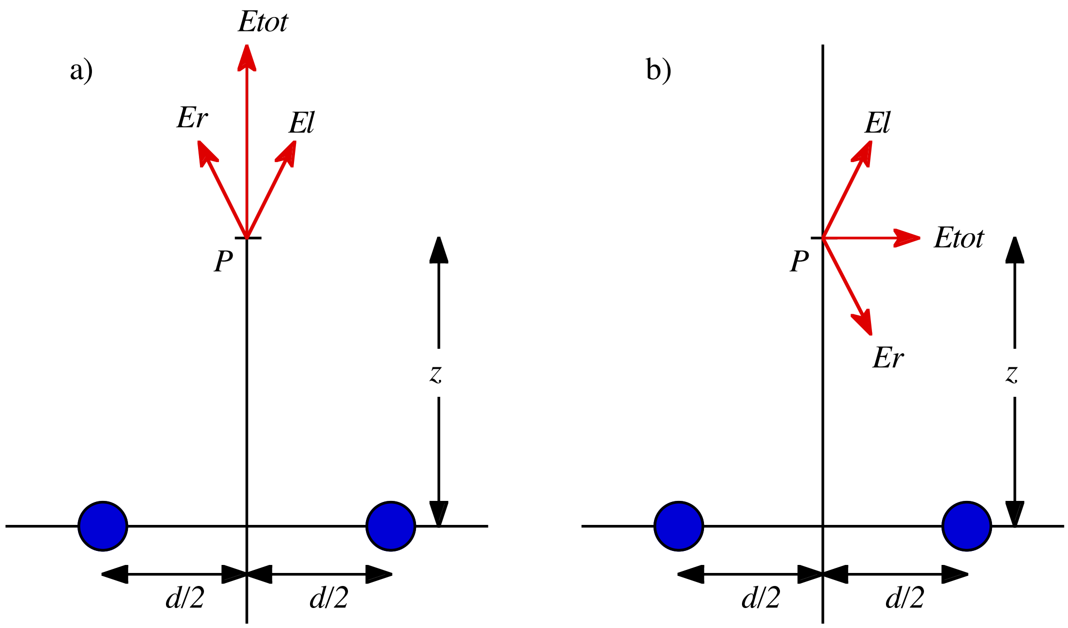

Figure 2. Problem 2

a) Figure 2a shows that the x components of the electric fields generated by the two point charges cancel. The total electric field at P is equal to the sum of the z components of the electric fields generated by the two point charges:

$$\overline{E}(\overline{P}) = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{1}{4}d^2 + z^2\right)}\frac{z}{\sqrt{\frac{1}{4}d^2 + z^2}}\hat{z} = \frac{1}{2\pi\varepsilon_0}\frac{qz}{\left(\frac{1}{4}d^2 + z^2\right)^{3/2}}\hat{z}$$

When $z \gg d$ this equation becomes approximately equal to

$$\overline{E}(\overline{P}) \cong \frac{1}{2\pi\varepsilon_0}\frac{q}{z^2}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{z^2}\hat{z}$$

which is the Coulomb field generated by a point charge with charge $2q$.

b) For the electric fields generated by the point charges of the charge distribution shown in Figure 2b the z components cancel. The net electric field is therefore equal to

$$\bar{E}(\bar{P}) = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{1}{4}d^2 + z^2\right)}\frac{\frac{d}{2}}{\sqrt{\frac{1}{4}d^2 + z^2}}\hat{x} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left(\frac{1}{4}d^2 + z^2\right)^{3/2}}\hat{x}$$

2.2. Divergence and Curl of Electrostatic Fields

The electric field can be graphically represented using field lines. The direction of the field lines indicates the direction in which a positive test charge moves when placed in this field. The density of field lines per unit area is proportional to the strength of the electric field. Field lines originate on positive charges and terminate on negative charges. Field lines can never cross since if this would occur, the direction of the electric field at that particular point would be undefined. Examples of field lines produced by positive point charges are shown in Figure 2.5.

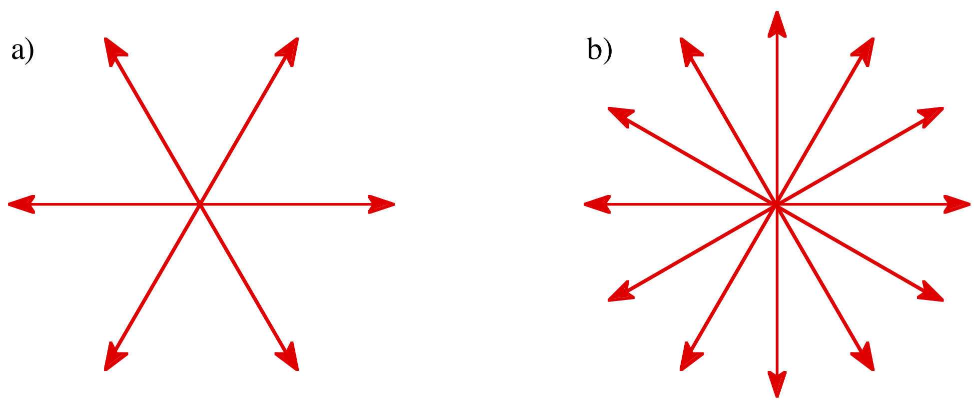

Figure a) Electric field lines generated by a positive point charge with charge *q*. b) Electric field lines generated by a positive point charge with charge 2*q*.

The flux of electric field lines through any surface is proportional to the number of field lines passing through that surface. Consider for example a point charge q located at the origin. The electric flux Φ_E through a sphere of radius r, centered on the origin, is equal to

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = \frac{1}{4\pi\varepsilon_0} \oint_{Surface} \left(\frac{q}{r^2} \hat{r} \right) \bullet \left(r^2 \sin\theta d\theta d\phi \hat{r} \right) = \frac{q}{\varepsilon_0}$$

Since the number of field lines generated by the charge q depends only on the magnitude of the charge, any arbitrarily shaped surface that encloses q will intercept the same number of field lines. Therefore the electric flux through any surface that encloses the charge q is equal to q/ε_0. Using the principle of superposition we can extend our conclusion easily to systems containing more than one point charge:

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = \sum_i \oint_{Surface} \bar{E}_i \bullet d\bar{a} = \frac{1}{\varepsilon_0} \sum_i q_i$$

We thus conclude that for an arbitrary surface and arbitrary charge distribution

$$\oint_{Surface} \bar{E} \bullet d\bar{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

where Q_{enclosed} is the total charge enclosed by the surface. This is called **Gauss's law**. Since this equation involves an integral it is also called **Gauss's law in integral form**.

Using the divergence theorem the electric flux Φ_E can be rewritten as

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = \int_{Volume} \left(\bar{\nabla} \bullet \bar{E} \right) d\tau$$

We can also rewrite the enclosed charge Q_{encl} in terms of the charge density ρ:

$$Q_{enclosed} = \int_{Volume} \rho d\tau$$

Gauss's law can thus be rewritten as

$$\int_{Volume} \left(\bar{\nabla} \bullet \bar{E} \right) d\tau = \frac{1}{\varepsilon_0} \int_{Volume} \rho d\tau$$

Since we have not made any assumptions about the integration volume this equation must hold for any volume. This requires that the integrands are equal:

$$\bar{\nabla} \bullet \bar{E} = \frac{\rho}{\varepsilon_0}$$

This equation is called **Gauss's law in differential form**.

Gauss's law in differential form can also be obtained directly from Coulomb's law for a charge distribution $\rho(\bar{r}')$:

$$\bar{E}(\bar{r}') = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\Delta\hat{r}}{(\Delta r)^2} \rho(\bar{r}') d\tau'$$

where $\Delta\bar{r} = \bar{r} - \bar{r}'$. The divergence of $\bar{E}(\bar{r})$ is equal to

$$\overline{\nabla} \bullet \overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \left(\overline{\nabla} \bullet \frac{\Delta\hat{r}}{(\Delta r)^2} \right) \rho(\overline{r}\,') d\tau\,' = \frac{1}{4\pi\varepsilon_0} \int_{Volume} 4\pi\delta^3(\overline{r} - \overline{r}\,')\rho(\overline{r}\,') d\tau\,' = \frac{\rho(\overline{r})}{\varepsilon_0}$$

which is Gauss's law in differential form. Gauss's law in integral form can be obtained by integrating $\overline{\nabla} \bullet \overline{E}(\overline{r})$ over the volume V:

$$\int_{Volume} \left(\overline{\nabla} \bullet \overline{E}(\overline{r}) \right) d\tau = \int_{Surface} \overline{E} \bullet d\overline{a} = \Phi_E = \int_{Volume} \frac{\rho(\overline{r})}{\varepsilon_0} d\tau = \frac{Q_{Enclosed}}{\varepsilon_0}$$

Example: Problem 2.42

If the electric field in some region is given (in spherical coordinates) by the expression

$$\overline{E}(\overline{r}) = \frac{A\hat{r} + B\sin\theta\cos\phi\hat{\phi}}{r}$$

where A and B are constants, what is the charge density ρ?

The charge density ρ can be obtained from the given electric field, using Gauss's law in differential form:

$$\rho = \varepsilon_0 \left(\overline{\nabla} \bullet \overline{E} \right) = \varepsilon_0 \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r \right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta E_\theta \right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \left(E_\phi \right) \right) =$$

$$= \varepsilon_0 \left(\frac{1}{r^2} \frac{\partial}{\partial r} (Ar) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \left(\frac{B\sin\theta\cos\phi}{r} \right) \right) = \varepsilon_0 \frac{A}{r^2} - \varepsilon_0 \frac{B}{r^2} \sin\phi$$

2.2.1. The curl of *E*

Consider a charge distribution $\rho(r)$. The electric field at a point P generated by this charge distribution is equal to

$$\overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\Delta\hat{r}}{(\Delta r)^2} \rho(\overline{r}\,') d\tau\,'$$

where $\Delta\overline{r} = \overline{r} - \overline{r}\,'$. The curl of $\overline{E}$ is equal to

$$\overline{\nabla} \times \overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int \left(\overline{\nabla} \times \frac{\Delta\hat{r}}{(\Delta r)^2} \right) \rho(\overline{r}\,') d\tau\,'$$

However, $\overline{\nabla} \times \hat{r} / r^2 = 0$ for every vector $\bar{r}$ and we thus conclude that

$$\overline{\nabla} \times \overline{E}(\bar{r}) = 0$$

2.2.2. Applications of Gauss's law

Although Gauss's law is always true it is only a useful tool to calculate the electric field if the charge distribution is symmetric:

1. If the charge distribution has **spherical symmetry**, then Gauss's law can be used with concentric spheres as Gaussian surfaces.

2. If the charge distribution has **cylindrical symmetry**, then Gauss's law can be used with coaxial cylinders as Gaussian surfaces.

3. If the charge distribution has **plane symmetr**y, then Gauss's law can be used with pill boxes as Gaussian surfaces.

Example: Problem

Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density ρ) of radius R.

The charge distribution has spherical symmetry and consequently the Gaussian surface used to obtain the electric field will be a concentric sphere of radius r. The electric flux through this surface is equal to

$$\Phi_E = \oint_{Surface} \overline{E} \bullet d\bar{a} = 4\pi r^2 E(r)$$

The charge enclosed by this Gaussian surface is equal to

$$Q_{Enclosed} = \frac{4}{3}\pi r^3 \rho$$

Applying Gauss's law we obtain for the electric field:

$$E(r) = \frac{1}{4\pi r^2}\frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{4\pi r^2}\frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0} = \frac{\rho}{3\varepsilon_0} r$$

Example: Problem

Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin: $\rho = k\, r$, for some constant k.

The charge distribution has spherical symmetry and we will therefore use a concentric sphere of radius r as a Gaussian surface. Since the electric field depends only on the distance r, it is constant on the Gaussian surface. The electric flux through this surface is therefore equal to

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = 4\pi r^2 E(r)$$

The charge enclosed by the Gaussian surface can be obtained by integrating the charge distribution between $r' = 0$ and $r' = r$:

$$Q_{Enclosed} = \int_{Volume} \rho(\bar{r}')d\tau = \int_0^r kr'\left(4\pi r'^2\right)dr' = \pi k r^4$$

Applying Gauss's law we obtain:

$$\Phi_E = 4\pi r^2 E(r) = \frac{Q_{Enclosed}}{\varepsilon_0} = \frac{\pi k r^4}{\varepsilon_0}$$

or

$$E(r) = \frac{\left(\frac{\pi k r^4}{\varepsilon_0}\right)}{4\pi r^2} = \frac{1}{4\varepsilon_0} k r^2$$

Example: Problem

A long coaxial cable carries a uniform (positive) volume charge density ρ on the inner cylinder (radius a), and uniform surface charge density on the outer cylindrical shell (radius b). The surface charge is negative and of just the right magnitude so that the cable as a whole is neutral. Find the electric field in each of the three regions: (1) inside the inner cylinder ($r < a$), (2) between the cylinders ($a < r < b$), (3) outside the cable ($b < r$).

The charge distribution has cylindrical symmetry and to apply Gauss's law we will use a cylindrical Gaussian surface. Consider a cylinder of radius r and length L. The electric field generated by the cylindrical charge distribution will be radially directed. As a consequence, there will be no electric flux going through the end caps of the cylinder (since here $\overline{E} \perp d\overline{a}$). The total electric flux through the cylinder is equal to

$$\Phi_E = \oint_{Surface} \overline{E} \bullet d\overline{a} = 2\pi r L E(r)$$

The enclosed charge must be calculated separately for each of the three regions:

1. $r < a$: $Q_{Enclosed} = \pi r^2 L\rho$
2. $a < r < b$: $Q_{Enclosed} = \pi a^2 L\rho$
3. $b < r$: $Q_{Enclosed} = 0$

Applying Gauss's law we find

$$E(r) = \frac{1}{2\pi r L}\frac{Q_{Enclosed}}{\varepsilon_0}$$

Substituting the calculated Q_{encl} for the three regions we obtain

1. $r < a$: $E(r) = \frac{1}{2\pi r L}\frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{2\pi r L}\frac{\pi r^2 L\rho}{\varepsilon_0} = \frac{1}{2\varepsilon_0} r\rho$.
2. $a < r < b$: $E(r) = \frac{1}{2\pi r L}\frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{2\pi r L}\frac{\pi a^2 L\rho}{\varepsilon_0} = \frac{1}{2\varepsilon_0}\frac{a^2}{r}\rho$
3. $b < r$ $E(r) = \frac{1}{2\pi r L}\frac{Q_{Enclosed}}{\varepsilon_0} = 0$

2.3. The Electric Potential

The requirement that the curl of the electric field is equal to zero limits the number of vector functions that can describe the electric field. In addition, a theorem discussed in Chapter 1 states that any vector function whose curl is equal to zero is the gradient of a scalar function. The scalar function whose gradient is the electric field is called the **electric potential *V*** and it is defined as

$$\overline{E} = -\overline{\nabla} V$$

Taking the line integral of $\overline{\nabla} V$ between point *a* and point *b* we obtain

$$\int_a^b \overline{\nabla} V \bullet d\bar{l} = V(b) - V(a) = -\int_a^b \overline{E} \bullet d\bar{l}$$

Taking *a* to be the reference point and defining the potential to be zero there, we obtain for *V*(*b*)

$$V(b) = -\int_a^b \overline{E} \bullet d\bar{l}$$

The choice of the reference point *a* of the potential is arbitrary. Changing the reference point of the potential amounts to adding a constant to the potential:

$$V'(b) = -\int_{a'}^b \overline{E} \bullet d\bar{l} = -\int_{a'}^a \overline{E} \bullet d\bar{l} - \int_a^b \overline{E} \bullet d\bar{l} = K + V(b)$$

where *K* is a constant, independent of *b*, and equal to

$$K = -\int_{a'}^a \overline{E} \bullet d\bar{l}$$

However, since the gradient of a constant is equal to zero

$$E' = -\overline{\nabla} V' = -\overline{\nabla} V = E$$

Thus, the electric field generated by V' is equal to the electric field generated by V. The physical behavior of a system will depend only on the difference in electric potential and is therefore independent of the choice of the reference point. The most common choice of the reference point in electrostatic problems is infinity and the corresponding value of the potential is usually taken to be equal to zero:

$$V(b) = -\int_{\infty}^{b} \overline{E} \bullet d\bar{l}$$

The unit of the electrical potential is the Volt (V, 1V = 1 Nm/C).

Example: Problem

One of these is an impossible electrostatic field. Which one?

a) $E = k\left[(xy)\hat{i} + (2yz)\hat{j} + (3xz)\hat{k}\right]$

b) $E = k\left[(y^2)\hat{i} + (2xy + z^2)\hat{j} + (2yz)\hat{k}\right]$

Here, k is a constant with the appropriate units. For the *possible* one, find the potential, using the origin as your reference point. Check your answer by computing $\overline{\nabla} V$.

a) The curl of this vector function is equal to

$$\overline{\nabla} \times \overline{E} = k\left(\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right)\hat{i} + k\left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3xz)\right)\hat{j} + k\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right) = k\left(-2y\hat{i} - 3z\hat{j} - x\hat{k}\right)$$

Since the curl of this vector function is not equal to zero, this vector function can not describe an electric field.

b) The curl of this vector function is equal to

$$\overline{\nabla} \times \overline{E} = k\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy + z^2)\right)\hat{i} + k\left(\frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(2yz)\right)\hat{j} + k\left(\frac{\partial}{\partial x}(2xy + z^2) - \frac{\partial}{\partial y}(y^2)\right) = 0$$

Since the curl of this vector function is equal to zero it can describe an electric field. To calculate the electric potential V at an arbitrary point (x, y, z), using $(0, 0, 0)$ as a reference point, we have to evaluate the line integral of $\overline{E}$ between $(0, 0, 0)$ and (x, y, z). Since the line integral of $\overline{E}$ is path independent we are free to choose the most convenient integration path. I will use the following integration path:

$$(0,0,0) \rightarrow (x,0,0) \rightarrow (x,y,0) \rightarrow (x,y,z)$$

The first segment of the integration path is along the x axis:

$$d\bar{l} = dx\hat{i}$$

and

$$\overline{E} \bullet d\bar{l} = ky^2 dx = 0$$

since $y = 0$ along this path. Consequently, the line integral of $\overline{E}$ along this segment of the integration path is equal to zero. The second segment of the path is parallel to the y axis:

$$d\bar{l} = dy\hat{j}$$

and

$$\overline{E} \bullet d\bar{l} = k\left(2xy + z^2\right)dy = 2kxydy$$

since $z = 0$ along this path. The line integral of $\overline{E}$ along this segment of the integration path is equal to

$$\int_{(x,0,0)}^{(x,y,0)} \overline{E} \bullet d\bar{l} = \int_0^y 2kxydy = kxy^2$$

The third segment of the integration path is parallel to the z axis:

$$d\bar{l} = dz\hat{k}$$

and

$$\overline{E} \bullet d\bar{l} = 2k\left(yz\right)dz$$

The line integral of $\overline{E}$ along this segment of the integration path is equal to

$$\int_{(x,y,0)}^{(x,y,z)} \bar{E} \bullet d\bar{l} = \int_0^z 2k(yz)dz = kyz^2$$

The electric potential at (x, y, z) is thus equal to

$$\begin{aligned} V(x,y,z) &= -\int_{(0,0,0)}^{(x,0,0)} \bar{E} \bullet d\bar{l} - \int_{(x,0,0)}^{(x,y,0)} \bar{E} \bullet d\bar{l} - \int_{(x,y,0)}^{(x,y,z)} \bar{E} \bullet d\bar{l} = \\ &= 0 - kxy^2 - kyz^2 = -k\left(xy^2 + yz^2\right) \end{aligned}$$

The answer can be verified by calculating the gradient of V:

$$\bar{\nabla} V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} = -k\left(y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + \left(2yz\right)\hat{k}\right) = -\bar{E}$$

which is the opposite of the original electric field $\bar{E}$.

The advantage of using the electric potential V instead of the electric field is that V is a scalar function. The total electric potential generated by a charge distribution can be found using the superposition principle. This property follows immediately from the definition of V and the fact that the electric field satisfies the principle of superposition. Since

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + ...$$

it follows that

$$V = -\int_\infty^b \bar{E} \bullet d\bar{l} = -\int_\infty^b \bar{E}_1 \bullet d\bar{l} - \int_\infty^b \bar{E}_2 \bullet d\bar{l} - \int_\infty^b \bar{E}_3 \bullet d\bar{l} - ... = V_1 + V_2 + V_3 + ...$$

This equation shows that the total potential at any point is the algebraic sum of the potentials at that point due to all the source charges separately. This ordinary sum of scalars is in general easier to evaluate then a vector sum.

Example: Problem

Suppose the electric potential is given by the expression

$$V(\bar{r}) = A\frac{e^{-\lambda r}}{r}$$

for all r (A and λ are constants). Find the electric field $\bar{E}(\bar{r})$, the charge density $\rho(\bar{r})$, and the total charge Q.

The electric field $\overline{E}(\overline{r})$ can be immediately obtained from the electric potential:

$$\overline{E}(\overline{r}) = -\overline{\nabla} V(\overline{r}) = -\frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right)\hat{r} = \left(\lambda A\frac{e^{-\lambda r}}{r} + A\frac{e^{-\lambda r}}{r^2}\right)\hat{r}$$

The charge density $\rho(\overline{r})$ can be found using the electric field $\overline{E}(\overline{r})$ and the following relation:

$$\overline{\nabla} \bullet \overline{E}(\overline{r}) = \frac{\rho(\overline{r})}{\varepsilon_0}$$

This expression shows that

$$\rho(\overline{r}) = \varepsilon_0\left[\overline{\nabla} \bullet \overline{E}(\overline{r})\right]$$

Substituting the expression for the electric field $\overline{E}(\overline{r})$ we obtain for the charge density $\rho(\overline{r})$:

$$\begin{aligned}
\rho(\overline{r}) &= \varepsilon_0 A\left[\overline{\nabla} \bullet \left((1+\lambda r)e^{-\lambda r}\frac{\hat{r}}{r^2}\right)\right] = \\
&= \varepsilon_0 A\left[(1+\lambda r)e^{-\lambda r}\left(\overline{\nabla} \bullet \frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2} \bullet \overline{\nabla}\left((1+\lambda r)e^{-\lambda r}\right)\right] = \\
&= \varepsilon_0 A\left[4\pi(1+\lambda r)e^{-\lambda r}\delta^3(\overline{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right] = \varepsilon_0 A\left[4\pi\delta^3(\overline{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right]
\end{aligned}$$

The total charge Q can be found by volume integration of $\rho(\overline{r})$:

$$\begin{aligned}
Q_{tot} = \int_{Volume} \rho(\overline{r})d\tau &= \int_0^\infty \varepsilon_0 A\left[4\pi\delta^3(\overline{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right]4\pi r^2 dr = \\
&= 4\pi\varepsilon_0 A\left[\int_0^\infty 4\pi\delta^3(\overline{r})r^2 dr - \int_0^\infty r\lambda^2 e^{-\lambda r}dr\right] = \\
&= -4\pi\varepsilon_0 A\int_0^\infty r\lambda^2 e^{-\lambda r}dr
\end{aligned}$$

The integral can be solved easily:

$$\int_0^\infty re^{-\lambda r}dr = -\frac{d}{d\lambda}\int_0^\infty e^{-\lambda r}dr = -\frac{d}{d\lambda}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^2}$$

The total charge is thus equal to

$$Q_{tot} = -4\pi\varepsilon_0 A$$

The charge distribution $\rho(\bar{r})$ can be directly used to obtained from the electric potential $V(\bar{r})$

$$\rho(\bar{r}) = \varepsilon_0\left[\overline{\nabla} \bullet \overline{E}(\bar{r})\right] = -\varepsilon_0\left[\overline{\nabla} \bullet \overline{\nabla} V(\bar{r})\right] = -\varepsilon_0 \overline{\nabla}^2 V(\bar{r})$$

This equation can be rewritten as

$$\overline{\nabla}^2 V(\bar{r}) = -\frac{\rho(\bar{r})}{\varepsilon_0}$$

and is known as **Poisson's equation**. In the regions where $\rho(\bar{r}) = 0$ this equation reduces to **Laplace's equation**:

$$\overline{\nabla}^2 V(\bar{r}) = 0$$

The electric potential generated by a discrete charge distribution can be obtained using the principle of superposition:

$$V_{tot}(\bar{r}) = \sum_{i=1}^{n} V_i(\bar{r})$$

where $V_i(\bar{r})$ is the electric potential generated by the point charge q_i. A point charge q_i located at the origin will generate an electric potential $V_i(\bar{r})$ equal to

$$V_i(\bar{r}) = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{q_i}{r'^2} dr' = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r}$$

In general, point charge q_i will be located at position $\bar{r}_i$ and the electric potential generated by this point charge at position $\bar{r}$ is equal to

$$V_i(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{\left|\bar{r} - \bar{r}_i\right|}$$

The total electric potential generated by the whole set of point charges is equal to

$$V_{tot}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\left|\bar{r} - \bar{r}_i\right|}$$