Outcome of Last Lecture (30/03/2020):

We discussed the dynamics of two body interaction and applied L.T. to investigate whether the principle of conservation of linear momentum was valid under such transformation. It was observed that the mathematical definition of linear momentum $\vec{p} = m\vec{u}$ (with $m = \text{constant}$) failed to satisfy the principle. But we know that L.T. is a good transformation and is consistent with the postulates of Special Theory of Relativity. We observed that if the same definition of linear momentum as mentioned above was retained with a modified definition of mass 'm' as a velocity dependent quantity $m = \frac{m_0}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$, then the crisis was over.

Henceforth we accept this new definition of Linear Momentum.

Then what about Newton's law of motion?

Let us try

$$\vec{F} = \frac{d(m\vec{u})}{dt} = \frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\right) \quad with\, \beta = \frac{u}{c} \qquad (15)$$ (Eqn no. is continued from the last lecture)

when the mass is a velocity dependent quantity as defined above.

Thus we get $\vec{F} = m\frac{d\vec{u}}{dt} + \frac{dm}{dt}\vec{u}$ …………………….. (16)

We now try a situation when the applied force changes the velocity of a particle of mass 'm' from '0' to 'u' during an interval of time. Hence the line integral of the force within the above mentioned limits of velocity will give the change of kinetic energy of the particle (obviously, initial kinetic energy is zero in this case). Thus

The change in K.E. =K = $\int_0^u \vec{F}.\,d\vec{l} = \int_0^u \left[m\frac{d\vec{u}}{dt}.\,d\vec{l} + \frac{dm}{dt}\vec{u}.d\vec{l}\right] = \int_0^u \left[m\vec{u}.d\vec{u} + u^2 dm\right]$ ……… (17)

Let us now try $m = \frac{m_0}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$ which gives $m^2 = \frac{m_0^2}{1-\frac{u^2}{c^2}}$, i.e.

$$m^2c^2 - m^2u^2 = m_0^2c^2 \qquad (18)$$

Now, $2mdmc^2 - 2u^2mdm - 2m^2\vec{u}.d\vec{u} = 0 \implies u^2dm + m\vec{u}.d\vec{u} = c^2dm$ ….……. (19)

Using equation (19) on the right side of equation (17), we get

$$K = c^2\int_0^u dm = c^2\int_{m_0}^{m} dm = c^2(m - m_0) \quad \text{......................} \quad (20)$$

NOTE:

1) When the particle is at rest, then $m = m_0$ and from eqn (20), the kinetic energy vanishes.

2) Also when it is in motion $K = mc^2 - m_0c^2$. Note that, with increase of velocity the first term on the r.h.s. increases (check), but the second term is a constant and hence K increases. And $K = mc^2 - m_0c^2 = \Delta mc^2$ shows that change of kinetic energy is related to change of inertial mass.

3) m_0c^2 has the dimension of energy; we call it the rest mass energy. Also $K + m_0c^2 = mc^2$, i.e. the l.h.s. is rest mass energy + kinetic energy which we say as the total energy and is equal to mc^2. We write $E = mc^2$.

4) If $u << c$, then

$$K = m_0c^2\left[\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right] = m_0c^2\left[\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} - 1\right] \quad \text{..............} \ (21)$$

$$= m_0c^2\left[1 - \left(-\frac{1}{2}\right)\frac{u^2}{c^2} + \frac{3}{8}\left(\frac{u^2}{c^2}\right)^2 + \ldots - 1\right] \approx \frac{1}{2}m_0u^2$$

Neglecting the higher order terms. Thus we drop to non-relativistic mechanics with the restriction $u << c$.

5) From equation (20), we see that $K \to \infty$ as $u \to c$, i.e. an infinite amount of work $\int \vec{F}.d\vec{l}$ is to be done on the particle to ac1celerate it to velocity 'c'. This again puts the restriction an upper limit on velocity.

<u>Relation Between Energy and Momentum:</u>

We start with $p = mu = \dfrac{m_0u}{\sqrt{1-\frac{u^2}{c^2}}}$ $\Longrightarrow$ $p^2c^2 - p^2u^2 = m_0^2u^2c^2$

$\Longrightarrow$ $u^2\left[m_0^2c^2 + p^2\right] = p^2c^2$ $\Longrightarrow$ $\dfrac{u^2}{c^2} = \dfrac{p^2}{p^2 + m_0^2c^2}$

Now from the relation $K + m_0c^2 = mc^2 = \dfrac{m_0c^2}{\left[1-\frac{u^2}{c^2}\right]^{\frac{1}{2}}} = \dfrac{m_0c^2}{\left[1-\frac{p^2}{p^2+m_0^2c^2}\right]^{\frac{1}{2}}}$

$$\Longrightarrow \quad (K + m_0c^2)^2 = \frac{m_0^2c^4\left(p^2 + m_0^2c^2\right)}{m_0^2c^2} = c^2\left(p^2 + m_0^2c^2\right) = p^2c^2 + m_0^2c^4$$

Hence the relation can be written as

$$\boxed{E^2 = p^2c^2 + m_0^2c^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad (22)$$

NOTE:

1) For a photon the rest mass is $m_0 = 0$, thus the momentum of the photon is $p = \frac{E}{c}$. If the frequency of the photon is ν, then $p = \frac{h\nu}{c} = \frac{h}{\lambda}$.

2) Equation (22) is useful to calculate the total energy if momentum is given or vice versa:

   Differentiating (22) w. r. t. p, $2E\frac{dE}{dp} = 2pc^2 \Rightarrow \frac{dE}{dp} = \frac{pc^2}{E} = \frac{pc^2}{mc^2} = \frac{p}{m} = u$ (23)

Problem: Prove that if $u << c$, the kinetic energy of a particle will be much less than its rest mass energy.

Hints:

➜ Express the relativistic K.E. as $K = (\gamma - 1)m_0c^2$

➜ Expand $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ and take the approximation for $u << c$ and give your argument.

Problem: Two lumps of clay each of rest mass m_0 move towards each other with equal speed $\frac{3}{5}c$ and stick together. What is the mass of the composite lump?

Solution:

Before collision After collision

If we consider the Centre of mass system, then the total momentum is zero before and after collision. (Try to understand this logic from the above diagrams).

m_0 is the mass of each lump before collision and M_0 is the rest mass of the composite lump.

Let mass of each lump is m when in motion, i.e. $m = \dfrac{m_0}{\sqrt{1-\beta^2}}$ and $\beta = \dfrac{3}{5}$ (given)

The total energy after collision is the rest mass energy, hence from the conservation of energy principle

$$M_0 c^2 = mc^2 + mc^2 = 2mc^2 = \frac{2m_0 c^2}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{2m_0 c^2 \times 5}{4} = \frac{5}{2} m_0 c^2$$

Thus mass of the composite lump is $M_0 = \dfrac{5}{2} m_0$

NOTE: The rest mass of the composite lump is not equal to $2m_0$, rather it is equal to $2.5m_0$.

STUDENTS,

LET US RETURN BACK TO THE DYNAMICS OF A PARTICLE. OUR DISCUSSION WAS SUSPENDED AFTER EQUATION (23). WE HAVE SHARED GOOD TIME AND HAVE GONE THROUGH TWO SIMPLE PROBLMS TO UNDERSTAND THE APPLICATION OF OUR PREVIOUS DISCUSSION.

ACCELERTION:

By definition, the acceleration of a particle is $\vec{a} = \dfrac{d\vec{u}}{dt}$.

Now $\quad \vec{F} = \dfrac{d\vec{p}}{dt} = \dfrac{d(m\vec{u})}{dt} = m\dfrac{d\vec{u}}{dt} + \vec{u}\dfrac{dm}{dt} = m\vec{a} + \vec{u}\dfrac{dm}{dt}$ ………………….. (24)

But $\quad m = \dfrac{E}{c^2}$ and hence

Kinetic energy

$$\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d\left(K + m_0 c^2\right)}{dt} = \frac{1}{c^2}\frac{dK}{dt} \quad \text{……………………} \quad (25)$$

Again $\; dK = \vec{F}.\, d\vec{l}$

Hence from equation (23), we write

$$\vec{F} = m\vec{a} + \vec{u}\frac{dm}{dt} = m\vec{a} + \frac{\vec{u}}{c^2}\frac{\vec{F}.\, d\vec{l}}{dt}$$

Thus acceleration can be expressed as $\boxed{\vec{a} = \frac{\vec{F}}{m} - \frac{\vec{u}}{mc^2}(\vec{F}.\vec{u})}$ [when $\vec{u} = \frac{d\vec{u}}{dt}$] …… (26)

IT IS VERY IMPORTNT TO NOTE FROM THE EQUATION (26) THAT THE ACCELERATION OF A PARTICLE IS NOT IN GENERAL PARALLEL TO THE APPLIED FORCE DUE TO THE PRESENCE OF THE SECOND TERM OF THE RIGHT HAND SIDE.

CASE I: When $\vec{F}$ and $\vec{u}$ are parallel, then from (26) $\vec{a}$ is parallel to both $\vec{F}$ and $\vec{u}$. This means the particle moves along a straight line parallel to $\vec{F}$ or $\vec{u}$. In this case the equation of motion can be written as

$$F = m\frac{du}{dt} + u\frac{dm}{dt} = \frac{m_0}{\sqrt{1-\beta^2}}\frac{du}{dt} + u\frac{d}{dt}\frac{m_0}{\sqrt{1-\beta^2}} = \frac{m_0 a}{\sqrt{1-\beta^2}} + um_0\frac{1}{(1-\beta^2)^{-\frac{3}{2}}}\left(-\frac{1}{2}\right)\left(\frac{-2u}{c^2}\right)\frac{du}{dt}$$

Or $$F = \frac{m_0 a}{\sqrt{1-\beta^2}} + \beta^2\frac{m_0 a}{(1-\beta^2)^{-\frac{3}{2}}} = \frac{m_0}{(1-\beta^2)^{-\frac{3}{2}}}a = m_{long}a \quad \ldots\ldots\ldots\ldots\ldots \quad (27)$$

The quantity $m_{long} = \frac{m_0}{(1-\beta^2)^{-\frac{3}{2}}}$ is called the "LONGITUDINAL MASS".

Example: Motion of a charged particle in a uniform Electric field.

CASE II: When $\vec{F}$ and $\vec{u}$ are perpendicular, then the 2nd term of the r. h. s. of equation (24) vanishes.

Hence from (24) $$F = \frac{m_0}{\sqrt{1-\beta^2}}a = m_{trans}a \quad \ldots\ldots\ldots. (28)$$

When $m_{trans} = \frac{m_0}{\sqrt{1-\beta^2}}$ is called "TRANSVERSE MASS"

Example: Force on a charged particle moving perpendicular to a magnetic field.

Problem:

Calculate the kinetic energy of an electron moving with a velocity of 0.98 times the velocity of light in the laboratory system.

Hints:

→ Use $K = mc^2 - m_0c^2$ and accept the standard rest mass of an electron.

Problem:

Show that the rest mass of a particle of momentum is ‘p’ and kinetic energy ‘K’ is

$$m_0 = \frac{p^2c^2 - K^2}{2Kc^2}$$

Hints:

Use Equation no. (22)

Problem:

If the total energy of a particle of mass ‘m’ is equal to twice its rest energy, then what will be the magnitude of its relativistic momentum?

Hints:

Use Equation no. (22)

Given that $E = 2m_0c^2$