STUDY MATERIAL (Sem – 4, STR) Dated: 14.04.2020

MINKOWSKI SPACE:

From the postulates of relativity, the speed of light remains same in every inertial frame, so the S and S' frames, we can write

$$x^2 + y^2 + z^2 = c^2t^2 \text{ and } x'^2 + y'^2 + z'^2 = c^2t'^2$$

And hence $$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

i.e. $$\sum_{\mu=1}^{4} x_\mu^2 = \sum_{\mu=1}^{4} x_\mu'^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

Note that $x_4 = ict$

[Equation (1a) is similar to the invariance of the square of the magnitude of a radius vector in 4 dimension under coordinate transformation]

Try to understand clearly that in non relativistic approach, time is taken as an absolute quantity, i.e. $t' = t$. That means it is assumed that t' does not depend on x, y, z under transformation. But in relativistic approach time is a quantity which depends on space coordinates under transformation. So in kinematics one has to introduce a fourth dimension and to keep dimensional homogeneity with the other space coordinates we propose the 4th coordinate as $ct = \omega$. We now write the L.T. in a symmetric way as

$$\left.\begin{aligned} x' &= \frac{x - \beta\omega}{\sqrt{1-\beta^2}}, \quad & x &= \frac{x' + \beta\omega}{\sqrt{1-\beta^2}}, \\ \omega' &= \frac{\omega - \beta x}{\sqrt{1-\beta^2}}, \quad & \omega &= \frac{\omega' + \beta x'}{\sqrt{1-\beta^2}} \end{aligned}\right\} \quad (2)$$

GEOMETRICAL REPRESENTATION:

Let us now try to understand the geometry of the space time as proposed above. For simplicity, let us assume a two dimensional diagram (suppressing '*y*' and '*z*' coordinates) spanned by x *and* $\omega(\omega = ct)$. Also we assume that the axes are orthogonal with the horizontal axis as the x-axis ($\omega = 0$, i.e. the axis along which the time t is constant) and vertical axis as the ω-axis (*x=0*).

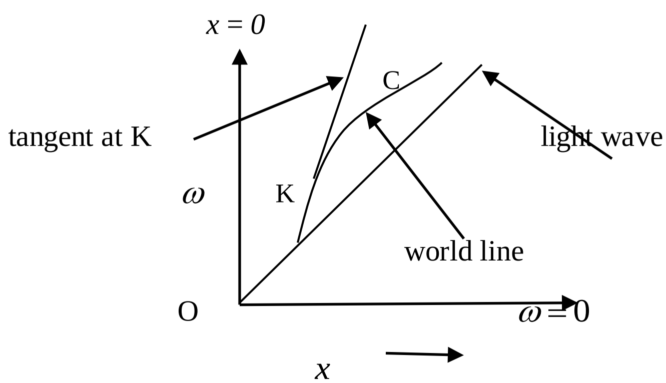

Fig. 1

The equation of the line bisecting the axes is $\boxed{x=\omega=ct}$, this line represents the propagation of light wave along the x-axis. Any other arbitrary curve (as indicated by KC) is called a world line in this space-time diagram. We consider such a world line KC. We draw a tangent at K on the world line which makes an angle θ (not shown in the diagram) with the ω-axis. Then

$$\tan\theta = \frac{dx}{d\omega} = \frac{1}{c}\frac{dx}{dt} = \frac{u}{c} \quad \text{……………} (3)$$

When θ is always less than 45^0 as $u < c$.

NOTE: Equation (3) shows that for a particle moving along the x-axis, the world line in the above diagram must lie between the lines *x=0* and $x=\omega=ct$.

LORENTZ TRANSFORMATION AND GEOMETRY:

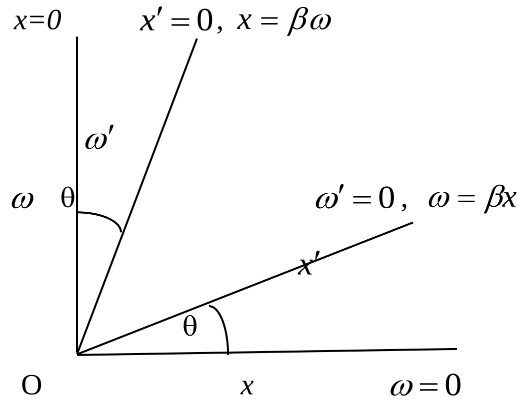

Fig. 2

Try to understand the situation when the S' frame moves along the common $x-x'$ axis with a uniform velocity, i.e. the origin of the S' frame moves. This statement can be expressed as $x'=0$, i.e. $x=\beta\omega$ [See equation (1)]. This is a straight line in the $x-\omega$ diagram and is drawn (see the line in Fig. 2). So this line is the ω'-axis ($x'=0$). From the equation $x=\beta\omega$, we have $\frac{dx}{d\omega}=\tan\theta=\beta$, thus the line is inclined at an angle $\theta=\tan^{-1}\theta$ with the ω axis. Similarly, for $\omega'=0$, i.e. $\omega=\beta x$ is a straight line in the $x-\omega$ diagram which is inclined at the same angle $\theta=\tan^{-1}\beta$ with the x-axis. Thus we get the $x'-\omega'$ diagram geometrically through L. T.

OBSERVE THE GEOMETRY OF L. T. IN FIG. 2. LORENTZ TRANSFORMATION CAUSES AN ORTHOGONAL SYSTEM TO A NON-ORTHOGONAL SYSTEM.

SIMULTANEITY:

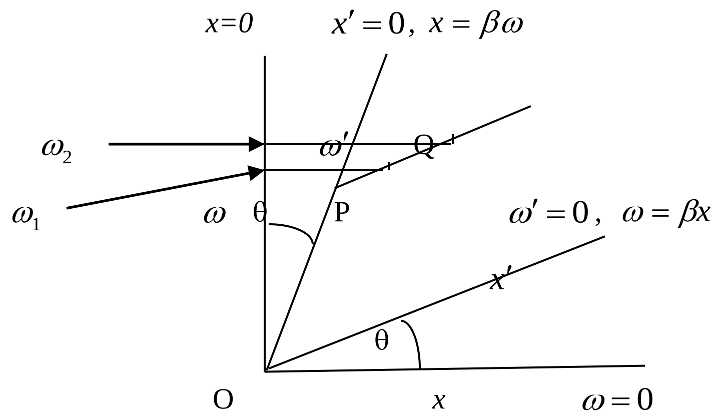

Fig. 3

If two events take place at same time, we call the events are simultaneous. If we consider a straight line parallel to the *x*-axis in the S - frame, then any two points as two events on that line represent different positions but time remains same (simultaneous). Also a vertical line parallel to the ω - axis represents a world line of particle at rest at a position as time flows. Similarly, the points on the world line parallel to the x' - axis represent simultaneous events in the S' - frame.

Note from Fig 3 that two events P and Q are simultaneous in the S' - frame, but take place at different times, say ω_1 and ω_2 in the S – frame. Similar conclusion can be drawn for simultaneous events in S – frame.

It is to be mentioned that a good explanation of Length Contraction and Time Dilation can be obtained from the Minkowski diagram, but for the time being that part is suspended from our discussion.

SPACE TIME INTERVAL:

We now consider two events in the S – frame described by the coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2).Then the interval is

$$S_{12} = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right]^{1/2} \quad \text{.............................. (4)}$$

For the S' frame moving with uniform velocity v along the common $x - x'$ axis, the same events take place at (x'_1, y'_1, z'_1, t'_1) and (x'_2, y'_2, z'_2, t'_2) and the interval is given as

$$S'_{12} = \left[c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2\right]^{1/2} \quad \text{.............................. (5)}$$

HOME WORK: Using L. T. show that S_{12}^2 and S'^2_{12} are equal.

CONCLUSION:

The square of the interval between two events in Minkowski space remains invariant under transformation from one inertial system to another inertial system.

Now consider the S- frame and an event point A say, as shown in the fig. 4. Now it is possible to find an inertial frame S' whose ω' axis is the straight line connecting the points O and A as shown. So we can say that all events on that straight line are at the same space point in that S' frame mentioned above. Because the ω' axis satisfies the equation $x' = 0$ as we have discussed earlier. Thus the event represented by the point A takes place at a later time than the event at O.

CONCLUSION:

The event at O takes place earlier than the event at A. Any point on the line OA represents a FUTURE event w. r. to O. Thus the shaded region in the upper half of the Fig. 5 represents "ABSOLUTE FUTURE".

Similar conclusion can be drawn for the shaded region below the x - axis the ABSOLUTE PAST, i.e. any point in that shaded region represents an event which occurred earlier than the event at O (Fig. 5).

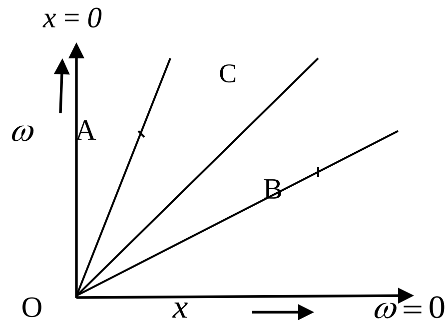

Fig. 4

Now we consider another point at B. As argued earlier we can choose OB as the x' - axis of a suitable S' - frame whose equation is $\omega' = 0$.

Note carefully that all events represented by points on the line OB are SIMULTANEOUS in the S' - frame. Obviously this region is called PRESENT (shown in Fig. 5).

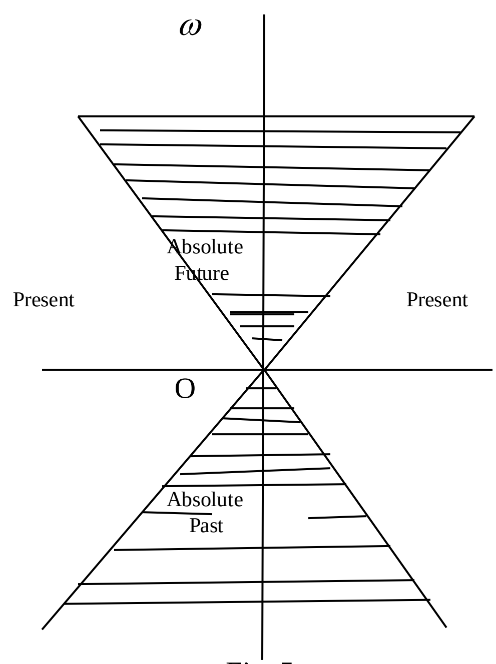

Fig. 5

A CRITICAL INVESTIGATION:

Recall equation no. 4 $S_{12} = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right]^{1/2}$

Case I:

If the events take place at the same position in the S' - frame, then we must have $x_2' = x_1'$, $y_2' = y_1'$ and $z_2' = z_1'$. Then from equation 4

$$S_{12}^2 = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right] = S_{12}'^2 = c^2(t_2' - t_1')^2$$

But $t_2' \rangle t_1'$ and hence $S_{12}^2 \rangle 0$. **Thus the space time interval is real.** Such an interval with

$$c^2(t_2 - t_1)^2 \rangle (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

is called TIME LIKE INTERVAL. The world lines in the shaded zone of Fig. 5 (Absolute Future and Past) are time like.

Case II:

If the events take place simultaneously in the S' frame (the PRESENT zone in Fig. 5) then $t_2' = t_1'$, then

$$S_{12}^2 = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right] = S_{12}'^2 = -\left[(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

and hence $S_{12}^2 \langle 0$. The space time interval is imaginary. Such an interval with

$$c^2(t_2 - t_1)^2 \langle (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

is called SPACE LIKE INTERVAL. The world lines in the PRESENT zone of Fig. 5 are space like.

VERY IMPORTANT (Source of confusion):

If in the above treatment we assume the convention

$$S_{12} = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2\right]^{1/2} \text{ (which is equally good)}$$

Then $S_{12}^2 \langle 0$ gives the TIME LIKE interval and $S_{12}^2 \rangle 0$ stands for the SPACE LIKE interval.

PROPER TIME:

Before handling the four vectors, it is now important to introduce a clear picture of Proper Time.

The Proper Time is the time recorded by the clock attached to a body tracing a world line. It is designated by the symbol τ. Let us consider a world line with initial point as the origin of the S – frame. We follow the coordinate notation $(x_0, \vec{r})$ when $x_0 = ct$. Its also clear that $x_4 = ix_0$ (see equation (1a)).

We have already discussed that $(dS)^2$ is an invariant scalar under coordinate transformation. Hence we can write

$$\Delta S = \left[c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\right]^{\frac{1}{2}} = (\Delta S') \text{ ....} \quad (6)$$

But for a clock attached to the body $\Delta x' = \Delta y' = \Delta z' = 0$ and $\Delta t' = \Delta \tau$ (as discussed in the last paragraph), we can write from equation (6)

$$\Delta S' = c\Delta\tau \text{ ....} \quad (7)$$

Hence
$$c\Delta\tau = \left[c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\right]^{\frac{1}{2}}$$

i.e.
$$\Delta\tau = \Delta t\left[1 - \frac{1}{c^2}\left\{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2\right\}\right]^{\frac{1}{2}} = \Delta t\left[1 - \frac{u^2}{c^2}\right]^{\frac{1}{2}} = \Delta t\sqrt{1-\beta^2} \text{ .....} \quad (8)$$

FOUR VECTOR:

It is noted that relativistic consideration of kinematics introduces a fourth dimension and an event point in the Minkowski Space is represented as $(x_0, \vec{r})$ or as $(\vec{r}, x_4)$, where $\vec{r}$ is the 3-vector and $x_4 = ix_0 = ict$.

We start with notation $(x_0, \vec{r}) \equiv (x^0, x^1, x^2, x^3)$ [The notations with superscripts will be clear in our future discussion on TENSOR]

FOUR VELOCITY:

We start with the zeroeth component of velocity u^0 when

$$u^0 = \frac{dx^0}{d\tau} = \frac{dx^0}{dt}\frac{dt}{d\tau} = \frac{d(ct)}{dt}\frac{1}{\sqrt{1-\beta^2}} = \frac{c}{\sqrt{1-\beta^2}}$$

$$u^1 = \frac{dx^1}{d\tau} = \frac{dx^1}{dt}\frac{dt}{d\tau} = \frac{u_x}{\sqrt{1-\beta^2}}, \quad \text{similarly,} \quad u^2 = \frac{u_y}{\sqrt{1-\beta^2}} \text{ and } u^3 = \frac{u_z}{\sqrt{1-\beta^2}} \text{ ....} \quad (9)$$

Thus the four velocity vector can be written as $u^i = (\gamma c, \gamma\vec{u})$

The scalar product (or square of magnitude) of four velocity with itself is

$$u^i u_i = (\gamma c)^2 - \gamma^2 \vec{u}.\vec{u} = \gamma^2(c^2 - u^2) = c^2\frac{1-\beta^2}{1-\beta^2} = c^2 \text{ ....} \quad (10)$$

Which is an invariant scalar and $u^i u_i \rangle 0$, i.e. four velocity is a TIME LIKE vector.

FOUR MOMENTUM:

From the previous treatment the four momentum is given by $p^i = (\gamma m_0 c, \gamma m_0 \vec{u}) = (mc, m\vec{u}) = \left(\frac{E}{c}, m\vec{u}\right)$ [using $\gamma m_0 = m$ and $E = mc^2$] p^i is TIME LIKE (Check)