**STATISTICS-Sem 2-Gen-Laws of Addition and Multiplication, Independent Events-
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The addition law of Probability

The sample space S is the set of all possible outcomes of a given experiment. Certain events A and B are subsets of S. We have already defined what are meant by P(A), P(B) and their complements in the particular case in which the experiment had equally likely outcomes. Events, like sets, can be combined to produce new events.
• $A \cup B$ denotes the event that event A or event B (or both) occur when the experiment is performed.
• $A \cap B$ denotes the event that both A and B occur together.
In this Section we obtain expressions for determining the probabilities of these combined events, which are written $P(A \cup B)$ and $P(A \cap B)$ respectively.

The Addition Law of Probability - Simple Case

If two events A and B are mutually exclusive then

$$P(A \cup B) = P(A) + P(B)$$

The Addition Law of Probability - General Case

If two events are A and B then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If $A \cap B = \emptyset$, i.e. A and B are mutually exclusive, then $P(A \cap B) = P(\emptyset) = 0$, and this general expression reduces to the simpler case. This rule can be extended to three or more events, for example:
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Example: Consider a pack of 52 playing cards. A card is selected at random. What is the probability that the card is either a diamond or a ten?

Solution: If A is the event {a diamond is selected} and B is the event {a ten is selected} then obviously $P(A) = 13/52$ and $P(B) = 4/52$. The intersection event $A \cap B$ consists of only one member - the ten of diamonds - which gets counted twice hence $P(A \cap B) = 1/52$. Therefore $P(A \cup B) = 13/52 + 4/52 - 1/52 = 16/52$.

Example: A bag contains 20 balls, 3 are coloured red, 6 are coloured green, 4 are coloured blue, 2 are coloured white and 5 are coloured yellow. One ball is selected at random. Find the probabilities of the following events.
(a) the ball is either red or green
(b) the ball is not blue
(c) the ball is either red or white or blue.

Solution: Note that a ball has only one colour, designated by the letters R, G, B, W, Y .
(a) $P(R \cup G) = P(R) + P(G) = 3/20 + 6/20 = 9/20$.
(b) $P(B') = 1 - P(B) = 1 - 4/20 = 16/20 = 4/5$.
(c) The complementary event is $G \cup Y$, $P(G \cup Y) = 6/20 + 5/20 = 11/20$.
Hence $P(R \cup W \cup B) = 1 - 11/20 = 9/20$

Example: The following people are in a room: 5 men aged 21 and over, 4 men under 21, 6 women aged 21 and over, and 3 women under 21. One person is chosen at random. The following events are defined: A = {the person is aged 21 and over}; B = {the person is under 21}; C = {the person is male}; D = { the person is female}. Evaluate the following:
(a) $P(B \cup D)$
(b) $P(A' \cap C')$

Solution: (a) $P(B \cup D) = P(B) + P(D) - P(B \cap D)$
$P(B) = 7/18$, $P(D) = 9/18$, $P(B \cap D) = 3/18$ $\therefore P(B \cup D) = 7/18 + 9/18 - 3/18 = 13/18$
(b) $P(A' \cap C')$
A' = {people under 21}
C' = {people who are female} $\therefore P(A' \cap C') = 3/18 = 1/6$

Example: A card is drawn at random from a deck of 52 playing cards. What is the probability that it is an ace or a face card (i.e. K, Q, J)?

Solution: F = {face card}
A = {card is ace}
$P(F) = 12/52$, $P(A) = 4/52$ $\therefore P(F \cup A) = P(F) + P(A) - P(F \cap A) = 12/52 + 4/52 - 0 = 16/52$

Example: In a single throw of two dice, what is the probability that neither a double nor a sum of 9 will appear?

Solution: D = {double is thrown}
N = {sum is 9}
$P(D) = 6/36$ (36 possible outcomes in an experiment in which all the outcomes are equally probable).
$P(N) = P\{(6 \cap 3) \cup (5 \cap 4) \cup (4 \cap 5) \cup (3 \cap 6)\} = 4/36$
$P(D \cup N) = P(D) + P(N) - P(D \cap N) = 6/36 + 4/36 - 0 = 10/36$
$P((D \cup N)') = 1 - P(D \cup N) = 1 - 10/36 = 26/36$

Multiplication Rule:

$P(A \text{ and } B) = P(A) * P(B|A)$
The probability of events A and B occurring can be found by taking the probability of event A occurring and multiplying it by the probability of event B happening given that event A already happened.

Example: A bag contains 3 pink candies and 7 green candies. Two candies are taken out from the bag with replacement. Find the probability that both candies are pink.

Solution: Let A = event that first candy is pink and B = event that second candy is pink.
$\rightarrow P(A) = 3/10$ …(i)

Since the candies are taken out with replacement, this implies that the given events A and B are independent.

→ P (B|A) = P (B) =3/10 …(ii)

Hence by the multiplication law we get,

P (A∩B) = P (A) * P (B|A)

→ P (A∩B) =3/10*3/10 [using (i) and (ii)]
=9/100= 0.09

Dependent Events: where what happens **depends on** what happened before, such as taking cards from a deck makes less cards each time.

Independent Event

The literal meaning of Independent Events is the events which occur freely of each other. The events are independent of each other. In other words, the occurrence of one event does not affect the occurrence of the other. The probability of occurring of the two events are independent of each other.

An event A is said to be independent of another event B if the probability of occurrence of one of them is not affected by the occurrence of the other.

Suppose if we draw two cards from a pack of cards one after the other. The results of the two draws are independent if the cards are drawn with replacement i.e., the first card is put back into the pack before the second draw. If the cards are not replaced then the events of drawing the cards are not independent.

Statistically, An event A is said to be independent of another event B, if the conditional probability of A given B, i.e, P(A | B) is equal to the unconditional probability of A. P(B) ≠ 0.

P(A | B) = P(A)

The term mutually exclusive should not be mixed with the term independent. The term mutually exclusive is related to the occurrence of an event. By independence of events, we mean the independence of probability of occurrence of events.

Example: You toss a coin and it comes up "Heads" three times ... what is the chance that **the next toss** will also be a "Head"?

The chance is simply ½ (or 0.5) just like ANY toss of the coin.

We can calculate the chances of two or more **independent** events by **multiplying** the chances.

Example: Probability of 3 Heads in a Row

For each toss of a coin a Head has a probability of 0.5:

0.5

$0.5 \times 0.5 = 0.25$ $\left(\text{or } \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\right)$

$0.5 \times 0.5 \times 0.5 = 0.125$ $\left(\text{or } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$

And so the chance of getting 3 Heads in a row is **0.125**

Example: Why is it unlikely to get, say, 7 heads in a row, when *each* toss of a coin has a ½ chance of being Heads?

Because we are asking two different questions:

Question 1: What is the probability of 7 heads in a row?

Answer: 1/2×1/2×1/2×1/2×1/2×1/2×1/2 = **0.0078125** (less than 1%)

Question 2: When **we have just got 6 heads** in a row, what is the probability that **the next toss** is also a head?

Answer: ½, as the **previous** tosses don't affect the next toss

Notation

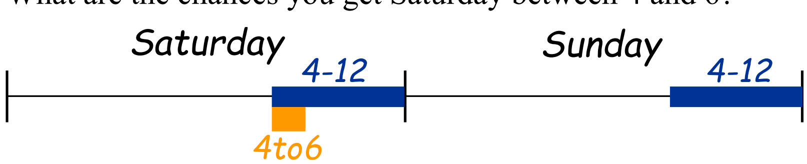
For Independent Events:

P(A and B) = P(A) × P(B)

Probability of A and B equals the probability of A times the probability of B

Example: your boss (to be fair) randomly assigns everyone an extra 2 hours work on weekend evenings between 4 and midnight.

What are the chances you get Saturday between 4 and 6?



Day: There are two days on the weekend, so **P(Saturday) = 0.5**

Time: you want the **2 hours** of "4 to 6", out of the **8 hours** of 4 to midnight):

P("4 to 6") = 2/8 = 0.25

And:

$$\textbf{P(Saturday and "4 to 6")} = \text{P(Saturday)} \times \text{P("4 to 6")}$$
$$= 0.5 \times 0.25$$
$$= \mathbf{0.125}$$

Or a 12.5% Chance

Example: the chance of a flight being delayed is 0.2 (=20%), what are the chances of no delays on a round trip

The chance of a flight **not** having a delay is 1 − 0.2 = **0.8**, so these are all the possible outcomes:

$0.8 \times 0.8 =$ **0.64** chance of **no delays**

$0.2 \times 0.8 =$ **0.16** chance of 1st flight delayed

$0.8 \times 0.2 =$ **0.16** chance of return flight delayed

$0.2 \times 0.2 =$ **0.04** chance of both flights delayed

When we add all the possibilities we get:

0.64 + 0.16 + 0.16 + 0.04 = **1.0**

They all add to 1.0, which is a good way of checking our calculations.

Result: **0.64**, or a 64% chance of no delays

Theorem 1

The events A and B are independent if $P(A \cap B) = P(A)\,P(B)$.

Proof: From the definition of an independent event, we have $P(A \mid B) = P(A) \Rightarrow P(A \cap B)/P(B) = P(A)$

or, $P(A \cap B) = P(A)\,P(B)$. Here, $P(B) \neq 0$.

Theorem 2

For two events A and B such that $P(A) \neq 0$, $P(B) \neq 0$. If A is independent of B, then B is independent of A.

Proof: If A is independent of B, we have

P(A | B) = P(A) or, P(A ∩ B) / P(B) = P(A) or, P(A ∩ B) = P(A) P(B) … (I)

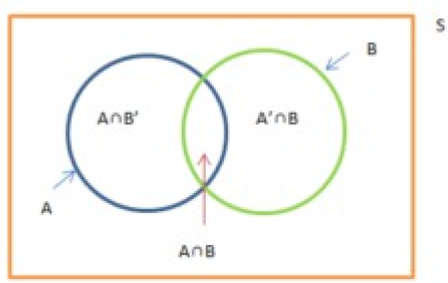
P(B|A) = P(A ∩ B) / P(A) = [P(A) P(B)] ⁄ P(A) = P(B) [from I]

So B is also independent of A.

Theorem 3

If A and B are independent events, then the events A and B’ are also independent.

Proof: The events A and B are independent, so, P(A ∩ B) = P(A) P(B).



From the Venn diagram, we see that the events A ∩ B and A ∩ B’ are mutually exclusive and together they form the event A.

A = (A ∩ B) ∪(A ∩ B’).
Also, P(A) = P[(A ∩ B) ∪(A ∩ B’)].
or, P(A) = P(A ∩ B) + P(A ∩ B’).

or, $P(A) = P(A) P(B) + P(A \cap B')$

or, P(A inter B') = $P(A) - P(A) P(B) = P(A) (1 - P(B)) = P(A) P(B')$

Mutually Independent Events

Three events A, B, and C are mutually independent if

$P(A \cap B) = P(A) P(B)$

$P(B \cap C) = P(B) P(C)$

$P(A \cap C) = P(A) P(C)$

$P(A \cap B \cap C) = P(A) P(B) P(C)$

Problem: Let A and B are two independent events such that P(A) = 0.2 and P(B) = 0.8. Find P(A and B), P(A or B), P(B not A), and P(neither A nor B).

Solution: Given P(A) = 0.2 and P(B) = 0.8 and events A and B are independent of each other.

P(A and B) = $P(A \cap B) = P(A) P(B) = 0.2 \times 0.8 = 0.16$.

P(A or B) = $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.8 - 0.16 = 0.84$.

P(B not A) = $P(B \cap A') = P(B) - P(A \cap B) = 0.8 - 0.16 = 0.64$.

And P(neither A nor B) = $P(A' \cap B') = 1 - P(A \cup B) = 1 - 0.84 = 0.16$.