Solution of a Non-linear Fredholm Integro-differential Equation by Adomian Decomposition Method (ADM) and Differential Transform Method (DTM)

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Abstract:

Nonlinear integral and integro-differential equations are usually hard to solve analytically and exact solutions are rather difficult to find out. There are many numerical and series solution methods are found in the literature. They are such as the Legendre wavelets method, the Haar function method, finite difference method, finite element method, the hybrid Legendre polynomials and block-pulse functions and many other semi-analytical methods. The Adomian decomposition method and recently developed differential Transform method are separately applied for the solution of nonlinear Fredholm integro-differential equation. Both the methods are used successfully and the exact solution of the problem is obtained.

Key words: Adomian Decomposition method(ADM), Adomian's polynomial, Differential Transform method(DTM), closed form solution, Integro-differential equation, Fredholm integral equation.

1.Introduction:

The objective of the decomposition method is to make possible physically realistic solutions of physical systems without the usual modeling and the solution compromises to achieve tractability. Since the beginning of the 1980s, **George Adomian[1-7]** has initially presented and developed a so-called decomposition method for solving linear or nonlinear problems such as ordinary and partial differential equations and also linear integral equations. Adomian's decomposition method (ADM) consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called **Adomian's polynomials**, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials. ADM is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation, and continuous with no resort to discretization and consequent computer-intensive calculations. Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian decomposition method **[1-7]**. A reliable modification of Adomian decomposition method has been done by **Wazwaz[8]**. Latter, the accuracy of Adomian decomposition method has been improved by **Jiao et al.[9]**. An efficient modification of ADM has been developed by **Luo [10]**. The decomposition method provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems **[1-5]**. This method efficiently works for initial-value or boundary-value problem and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. Moreover, we have the advantage of a single global method for solving ordinary or partial differential equations as well as many other types of equations such as integral equation, integro-differential equations.The solution of extraordinary differential equation has been obtained through Adomian decomposition method by the researchers in**[12-13]**. The application of Adomian decomposition method for the solution of linear and nonlinear Volterra-Fredholm integro-differential equations has also been established by **Inc. and Cherrualt[11]**.

Kaya et al[14-20] have successfully applied decomposition method to different types of linear and nonlinear equations, viz. coupled Viscous Burgers' equation, generalized nonlinear Boussinesq equations and generalized Fifth Order KdV equations, coupled Schrödinger-KdV equation, Kadomtsev-Petviashvili equation, generalized KdV and RLW equations, Sine-Gordon equation, generalized Hirota-Satsuma coupled KdV equation, generalized Regularized Long-Wave equation, KdV equation, generalized Burgers-Fisher equation, compound KdV–type and compound KdV-

Burgers type equations, Klein-Gordon equation, seven-order Sawada-Kotara equations, (2+1)-dimensional Boussinesq equations and (3+1)-dimensional KP equation, system of two dimensional Burgers' equations, two dimensional Sine-Gordon equations, Coupled higher-dimensional Burgers' equations, Whitham-Broer-Kaup equations, combined KdV-MKdV equation, modified KdV equation, Lienard equation, NLS equation and generalized modified Boussinesq equation, etc.

The convergence of Adomian decomposition method and the application of this method non- linear integral equations, evolution model, Lorenz's equation, Cauchy Problem, diffusion-convection problem, fourth-order semi-linear diffusion equation, Wave equations, Boundary value problems, Kortweg-de Vries equation and linear and nonlinear Volterra-Fredholm integro-differential equations have been established by notable researchers **Cherrualt, Abbaoui, Inc. and others[21-28]**.In this connection, it is worthwhile to mention that the lot of valuable research works by applying Adomian decomposition method are available from the notable works of **Wazwaz [29-33]**.

The decomposition method does not change the problem into a convenient one for the use of linear theory. It, therefore, provides more realistic solutions. It provides series solutions which generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. The method makes unnecessary the massive computations arising in the use of discretized methods for solution of partial differential equations and integral equations. No linearization or perturbation is required. It provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems. There are some quite significant advantages over other method which must assume linearity, "smallness", deterministic behavior, stationarity, restricted kind of stochastic behavior, uncoupled boundary conditions, etc. The method has features in common with many other methods, but it is distinctly different on close examination and one should not be misled by apparent simplicity into superficial conclusions.

To justify the ADM,we have applied another newly introduced method, known as **Differential Transform Method** (DTM) in this connection.

The classical Taylor series method is one of the earliest techniques to many problems, especially ordinary differential equations. However, since it requires a lot of symbolic calculation for the derivative of the function, it takes a lot of computational time. Here we introduce the updated version of the Taylor series method which is called differential transform method (DTM).The concept of differential transform method (DTM) was introduced by **Zhou[34]** (1986). This scheme is based on the Taylor series expansion to construct analytical solutions in the form of a polynomial by means of an iterative procedure. Recently researchers have applied the DTM to obtain analytical solutions for linear and nonlinear differential equations such as in two point boundary value problem by **Chenand and Liu[35]**, in the KdV and MKdV equations by **Angalgil and Ayaz[36]**, in the nonlinear parabolic-hyperbolic partial differential equations by **Biazar et al.[37],** inthe two-dimensional nonlinear Gas dynamic and Klien-Gordon equations by **Jafari et al.,[38]**. Differential transform method (DTM) is also applicable for solving differential as well integral equations and integro-differential equations both in linear and nonlinear forms.

2.Mathematical Aspect of Adomian Decomposition Method(ADM):

Consider the Integro-differential equation in the form

$$L[u(x)] = f(x) + N[u(x)], \qquad \ldots\ldots\ldots\ldots(1)$$

where L is a linear differential operator, $f(x)$ is a known function of x and $N[u(x)]$ is a non-linear integral operator.Adomian decomposition method defines the unknown function u(x) by an infinite series

$$u(x) = \sum_{i=0}^{\infty} u_i(x), \qquad \ldots\ldots\ldots\ldots(2)$$

where the components $u_i(x)$ are to be determined by using recurrence relations.

Define $v(x) = \sum_{k=0}^{\infty} \lambda^k . u_k(x),\ \lambda\ being\ a\ parameter\ and\ then\ v(x) = u(x)\ for\ \lambda = 1.$

The non-linear part $N[u(x)]$ can be decomposed in an infinite series of Adomian polynomials given by

$$N\,[u(x)] = \sum_{i=0}^{\infty} A_i\,(u_0, u_1 \,....\, u_i). \quad .... (3)$$

where $A_i's$ are the **Adomian polynomial** defined by

$$A_i(u_0\,, u_1, ...\ u_i\ ) = [\frac{1}{i!}\cdot\frac{d^i}{d\lambda^i}\{N(v(x)\,)\}]_{\lambda=0}, \qquad for\ i = 0,1,2, \cdots$$

$$= [\tfrac{1}{i!}\cdot\tfrac{d^i}{d\lambda^i}\{\,N(\textstyle\sum_{k=0}^{\infty} \lambda^k . u_k(x)\,)\}]_{\lambda=0,\ for\ i=0,1,2,...} \quad .... (4)$$

From (1) after using (2) and (3) one can write,

$$\sum_{i=0}^{\infty} u_i\,(x) = L^{-1}(f(x) + L^{-1}\left(\sum_{i=0}^{\infty} A_i\,(u_0, u_1 \,....\ u_i)\right) .... 4(a)$$

$$Proper\ choice\ of\ first\ few\ terms u_0(x), u_1(x)\ etc. can\ lead\ to\ a\ exact\ solution$$

$$of\ the\ nonlineal\ differential\ equation\ or\ nonlinear\ integro - differential\ equation.$$

3. Mathematical Aspect of Differential Transform Method (DTM):

The basic definition and the fundamental theorems of the differential transform and its applicability for various kinds of differential and integral equations are given in the references [34-38].The transformation of the kth derivative of a function u(x) in one variable is denoted by U_k and defined by

$$U_k = \frac{1}{k!}\frac{d^k}{dx^k}\,u(x)\bigg|_{x=x_0} \qquad .... (5)$$

and the inverse transformation is denoted by u(x) and defined by

$$u(x) = \sum_{k=0}^{\infty} U_k (x - x_0)^k \qquad ...... (6)$$

The following theorems can be deduced from (5) and (6), assuming$u(x), f(x)\ and\ g(x) \in C^{n+k}. That\ is\ u(x), f(x)\ and\ g(x) are\ continuously\ differentiable\ functions\ upto\ order$ $(n + k)$. Denote nth order derivative of $f(x)$ by $f^{(n)}(x)$.

$\boldsymbol{Theorem\ 1}$: $If\ u(x) = \lambda_1 f(x) \pm \lambda_2 g(x)\ , then\ U_k = \lambda_1 F_k \pm \lambda_2 G_k\ ,$
$where\ \lambda_1\ and\ \lambda_2\ \ are\ constants.$

$\boldsymbol{Theorem\ 2}$: $If\ \ u(x) = \lambda f^{(n)}(x)\ , then\ U_k =\ \lambda F_{n+k}\dfrac{(k+n)!}{k!}\ .$

$\boldsymbol{Theorem\ 3}$: $If\ \ u(x) = \lambda f(x)g(x)\ , then\ U_k =\ \lambda \sum_{j=0}^{k} F_j\, G_{k-j}$

$\boldsymbol{Theorem\ 4}$: $If\ u(x) = \lambda(x - x_0)^m\ , then\ \ U_k = \lambda\delta(k-m)\ , where$

$$\delta(k-m) = \begin{cases} 1\ , & k = m \\ 0\ , & k \neq m \end{cases} \quad (\ m \geq 0\ \ is\ an\ integer.)$$

$\boldsymbol{Theorem\ 5}$: $If\ u(x) =\ g_1(x)g_2(x) \ldots.. g_{n-1}(x)g_n(x)\ \ , then$

$$U_k = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots. \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1) \ldots. G_{n-1}(k_{n-1} - k_{n-2})G_n(k - k_{n-1}),$$

$$where\ G_i(k_i - k_{i-1}) = \frac{1}{(k_i - k_{i-1})!}.\frac{d^{(k_i - k_{i-1})}}{dx^{(k_i - k_{i-1})}} g_i(x)\ ,\ k_0 = 0\ and\ k_n = k\ .$$

$\boldsymbol{Theorem\ 6}$: $If\ \ u(x) = \int_{x_0}^{x} g(t)\, dt\ \ then\ U_k = \dfrac{G_{k-1}}{k}\ , \qquad k \geq 1\ .$

$\boldsymbol{Theorem\ 7}$: $If\ u(x) = \int_{x_0}^{x} g_1(t)g_2(t) \ldots.. g_n(t)dt\ \ , then$

$$U_k = \frac{1}{k} \sum_{k_{n-1}=0}^{k-1} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1) \ldots. G_{n-1}(k_{n-1} - k_{n-2})G_n(k - k_{n-1} - 1).$$

For solving Fredholm integral equation, the following relation is useful and can be derived from Theorem 7 and relation (6),

$$\int_a^b g_1(t)g_2(t) \ldots.. g_{n-1}(t)g_n(t)dt$$

$$= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}[((b - x_0)^k - (a - x_0)^k)] \times \left[\sum_{k_2=0}^{k-1} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1)G_3(k - k_2 - 1) \right] \right\}$$

Differential transform for some basic functions:

$(i)\ If\ u(x) = e^{\lambda x}, then\ U_k = \dfrac{\lambda^k}{k!}\ .$

$(ii)\ If\ u(x) = sin(ax+b), then\ U_k = \frac{a^k}{k!}\ sin\left(\frac{\pi x}{2}+b\right).$

$(iii)\ If\ u(x) = cos(ax+b), then\ U_k = \frac{a^k}{k!}\ cos\left(\frac{\pi x}{2}+b\right).$

4. Solution of the non-linear Fredholmintegro-differential equation by ADM

$$u''(x) = \frac{7}{3}cosx + \frac{\pi}{2}sinx + \frac{1}{2}\int_0^{\pi} sin(x-t)u^2(t)dt, \qquad \text{......} (7)$$

subject to the initial conditions $u(0) = 0, u'(0) = 0$.

Here the known function is given by

$$f(x) = \frac{7}{3}cosx + \frac{\pi}{2}sinx \quad \text{and} \quad N[u(x)] = \frac{1}{2}\int_0^{\pi} sin(x-t)u^2(t)dt \qquad \text{......} (8)$$

We can derive Adomian polynomials (using (4)) as

$$A_0 = \frac{1}{2}\int_0^{\pi} sin(x-t)[\,u_0^2(t)]dt$$

$$A_1 = \frac{1}{2}\int_0^{\pi} sin(x-t)\,[2u_0(t)u_1(t)]dt$$

$$A_2 = \frac{1}{2}\int_0^{\pi} sin(x-t)\,[2u_0(t)u_2(t) + u_1^2(t)]dt$$

$$A_3 = \frac{1}{2}\int_0^{\pi} sin(x-t)\,[2u_0(t)u_3(t) + 2u_1(t)u_2(t)]dt, \qquad etc. \qquad \text{......} (9)$$

For the equation in (7), the linear operator $L \equiv \frac{d^2}{dx^2}$. Operating L^{-1} on both sides of (7) and using initial condition, one gets

$$u(x) = L^{-1}\left[\frac{7}{3}cosx + \frac{\pi}{2}sinx\right] + L^{-1}\frac{1}{2}\int_0^{\pi} sin(x-t)[\,u^2\ (t)]dt$$

$$= \frac{7}{3}(1-cosx) + \frac{\pi}{2}(x-sinx) + \frac{1}{2}L^{-1}\int_0^{\pi} sin(x-t)[\,u^2\ (t)]dt$$

Using $u = \sum_{i=0}^{\infty} u_i\,(x)$ and $N(u(x)) = \sum_{i=0}^{\infty} A_i\,(u_0, u_1 \ldots.\ u_i)$; and equation (8), one gets

$$\sum_{i=0}^{\infty} u_i\,(x) = \frac{7}{3}(1 - cosx) + \frac{\pi}{2}(x - sinx) + L^{-1}\sum_{i=0}^{\infty} A_i(u_0, u_1 \,....\, u_i)$$

…………… 9(a)

We consider $u_0(x) = k(1 - cosx)$, where k is a constant with

$$u_0(0) = u_0^{'}(0) = 0.\ Now\ defineu_1(x)\ by$$

$$u_1(x) = -u_0(x) + \frac{7}{3}(1 - cosx) + \frac{\pi}{2}(x - sinx) + L^{-1}A_0(u_0), \qquad .....(10)$$

$$where, A_0(u_0) = \frac{1}{2}\int_0^{\pi} sin(x - t)[\,u_0^2\,(t)]dt = \frac{1}{2}\left[\mathrm{k}^2 \int_0^{\pi} sin(x - t)(1 - cost)^2\,\mathrm{dt}\right]$$

$$= k^2\left[-\frac{4}{3}cosx - \frac{\pi}{2}sinx\right]$$

$$Then, u_1(x) = -k(1 - cosx) + \frac{7}{3}(1 - cosx) + \frac{\pi}{2}(x - sinx) + L^{-1}A_0(u_0)$$

$$and\ \ L^{-1}A_0(u_0) = \mathrm{k}^2\left[-\frac{4}{3}(1 - cosx) - \frac{\pi}{2}(x - sinx)\right]$$

Therefore,$u_1(x) = \left(\frac{7}{3} - k - \frac{4}{3}k^2\right)(1 - cosx) + (1 - k^2)\frac{\pi}{2}(x - sinx)$.

We choose k=1 and one gets $u_1(x) \equiv 0, \forall x$ and then $\mathrm{u}_0(\mathrm{x}) = 1 - \cos \mathrm{x}$

… … … . . (11)

Now, we can develop a recurrence relations from 9(a) and (10) as

$$u_i = L^{-1}\big(A_{i-1}(u_{0,}u_{1,}\,.....,u_{i-1})\big), \qquad for\ i = 2, 3, \dots\dots (12)$$

It can be shown that all the $u_i(x)(for\ n \geq 2)$ *can be derived from* (12)

in the form of recurrence relation and all of them become zero

$$i.e., \qquad u_i(x) \equiv 0, \forall x\ and\ for\ i = 2,3,4, \dots\dots$$

Therefore we have derived the exact (closed form) solution of **the non-linear Fredholm integro-differential equation** (7) as $u(x) = \sum_{i=0}^{\infty} u_i\ (x) = \ u_1\ (x) = 1 - \cos(x)$.

5. Solution of the non-linear Fredholmintegro-differential equation by DTM

$$u^{''}(x) = \frac{7}{3}cosx + \frac{\pi}{2}sinx + \frac{1}{2}\int_0^{\pi} sin(x - t)u^2\,(t)dt, \qquad \dots\dots\dots (13)$$

subject to the initial conditions $u(0) = 0\,, u^{'}(0) = 0$.

Taking kth order derivative on both sides of (13) then dividing by $k!$ and putting $x = 0$,one gets

$$(k+1)(k+2)U_{k+2} = \frac{7}{3}\frac{1}{k!}cos\left(\frac{k\pi}{2}+x\right)\Big|_{x=0} + \frac{\pi}{2}\frac{1}{k!}sin\left(\frac{k\pi}{2}+x\right)\Big|_{x=0}$$

$$+\frac{1}{2}\frac{1}{k!}\int_0^\pi sin\left(\frac{k\pi}{2}+x-t\right)\Big|_{x=0} u^2(t)\,dt$$

$$Or, U_{k+2} = \frac{7}{3}\frac{1}{(k+2)!}cos\frac{k\pi}{2} + \frac{\pi}{2}\frac{1}{(k+2)!}sin\frac{k\pi}{2}$$
$$+\frac{1}{2}\frac{1}{(k+2)!}\int_0^\pi sin\left(\frac{k\pi}{2}-t\right)u^2(t)dt \qquad \ldots\ldots.\ldots (14)$$

$$Now, \int_0^\pi sin\left(\frac{k\pi}{2}-t\right)u^2(t)dt = \begin{cases}(-1)^{\frac{k-1}{2}}\int_0^\pi cost.u^2(t)dt\,, when\,, & k = odd\ integer\\ (-1)^{\frac{k-2}{2}}\int_0^\pi sint.u^2(t)dt\,, when\,, & k = even\ integer\end{cases}$$

$$Let\ \alpha = \int_0^\pi cost.u^2(t)dt\ and\ \beta = \int_0^\pi sint.u^2(t)dt$$

$$then\ \int_0^\pi sin\left(\frac{k\pi}{2}-t\right)u^2(t)dt = \begin{cases}(-1)^{\frac{k-1}{2}}\alpha\,, & for\ k = odd\ integer\\ (-1)^{\frac{k-2}{2}}\beta\,, & for\ k = even\ integer\end{cases}$$

<u>When k = odd integer, from (14), one gets</u>

$$U_{k+2} = \frac{\pi}{2}\frac{1}{(k+2)!}(-1)^{\frac{k-1}{2}} + \frac{1}{2(k+2)!}(-1)^{\frac{k-2}{2}}\alpha\ , \text{for } k = 1,3,5,\ldots \qquad \ldots(15)$$

<u>and when k = even integer, from (14), we get</u>

$$U_{k+2} = \frac{7}{3}\frac{1}{(k+2)!}(-1)^{\frac{k}{2}} + \frac{1}{2(k+2)!}(-1)^{\frac{k+2}{2}}\beta \quad \text{for } k = 0,2,4\ \ldots. \qquad \ldots (16)$$

We have, $U_0 = 0$ and $U_1 = 0$, from definition. Now from (15) and (16) we get,

for k=0: $U_2 = \frac{7}{6} - \beta \Rightarrow \beta = 4\left(\frac{7}{6} - U_2\right)$

for k= 1: $U_3 = \frac{\pi}{12} + \frac{\alpha}{12} \Rightarrow \alpha = 12\left(U_3 - \frac{\pi}{12}\right)$.

For k=2: $U_4 = -\frac{7}{3(4!)} + \frac{1}{2(4!)}\beta = -\frac{7}{3(4!)} + \frac{1}{2(4!)}4\left(\frac{7}{6} - U_2\right) \Rightarrow U_4 = -\frac{1}{2(3!)}U_2$

For k=3: $U_5 = \frac{\pi}{2(5!)} + \frac{1}{2(5!)}\alpha = \frac{\pi}{2(5!)} + \frac{1}{2(5!)}(12U_3 - \pi) \Rightarrow U_5 = \frac{12}{2(5!)}U_3$

When $k =$ odd integer:

$$U_{k+2} = \frac{\pi}{2}\frac{1}{(k+2)!}(-1)^{\frac{k-1}{2}} + \frac{1}{2(k+2)!}(-1)^{\frac{k-2}{2}}12\left(U_3 - \frac{\pi}{12}\right) \Rightarrow U_{k+2} = (6U_3)\frac{.(-1)^{\frac{k-2}{2}}}{(k+2)!}$$

$$\text{i.e. } U_{k+2} = (6U_3)\frac{.(-1)^{\frac{k-2}{2}}}{(k+2)!}\ for\ k = 1,3,5 \ldots..$$

$$This\ implies, \quad U_k = (6U_3)\frac{(-1)^{\frac{k+1}{2}}}{(k)!} for\ k = 3,5,7 \ldots.. \qquad \ldots..(17)$$

When k =even integer:

$$U_{k+2} = \frac{7}{3}\frac{1}{(k+2)!}(-1)^{\frac{k}{2}} + \frac{1}{2(k+2)!}(-1)^{\frac{k+2}{2}}4\left(\frac{7}{6} - U_2\right)$$

$$\Rightarrow U_{k+2} = (2U_2)\frac{(-1)^{\frac{k}{2}}}{(k+2)!}$$

$$\text{i.e.} \quad U_{k+2} = (2U_2)\frac{(-1)^{\frac{k}{2}}}{(k+2)!}\ for\ k = 2,4,6 \ldots\ldots$$

$$This\ gives\ us,\ U_k = (2U_2)\frac{(-1)^{\frac{k-2}{2}}}{(k)!} \quad for\ k = 2,4,6\ \ldots \qquad \ldots..(18)$$

$$\text{then } u(x) = \sum_{k=0}^{\infty} U_k x^k = \sum_{k=0(2)}^{\infty} U_k.x^k + \sum_{k=1(2)}^{\infty} U_k.x^k, (\text{since } U_0 = U_1 = 0)$$

$$= (2U_2)\sum_{k=2(2)}^{\infty}\frac{(-1)^{\frac{k-2}{2}}}{(k)!}x^k + (6U_3)\sum_{k=3(2)}^{\infty}\frac{(-1)^{\frac{k+1}{2}}}{(k)!}x^k$$

$$\therefore u(x) = 2U_2(1 - cosx) + 6U_3(x - sinx) \qquad \ldots\ldots(19)$$

Using the relations (17), (18), (19); and

$$\alpha = \int_0^{\pi} cost.u^2(t)dt\ and\ \beta = \int_0^{\pi} sint.u^2(t)dt,$$

one gets,

$$u^2(x) = 4U_2^2(1 - cosx)^2 + 36U_3^2(x - sinx)^2 + 24U_2U_3(1 - cosx)(x - sinx)$$

$$\therefore \alpha = \int_0^{\pi} cost.u^2(t)dt$$

$$= 4U_2^2 \int_0^{\pi} (1 - cost)^2 cost\, dt + 36U_3^2 \int_0^{\pi} (t - sint)^2\, cost\, dt$$
$$+ 24U_2U_3 \int_0^{\pi} (1 - cost)(t - sint)\, cost\, dt$$

$$and\ \ \beta = \int_0^{\pi} sint\, u^2(t)dt$$

$$= 4U_2^2 \int_0^{\pi} (1 - cost)^2 sintdt + 36U_3^2 \int_0^{\pi} (t - sint)^2\, sint\, dt$$
$$+ 24U_2U_3 \int_0^{\pi} (1 - cost)(t - sint)\, sint\, dt$$

$$Then\ \ \alpha = -4U_2^2\pi - 36U_3^2\left(\frac{3\pi}{2}\right) - 24U_2U_3\left(\frac{\pi^2}{4} + \frac{4}{3}\right) \ldots\ldots (20)$$

and $$\beta = 4U_2^2\left(\frac{8}{3}\right) - 36U_3^2\left(\frac{\pi^2}{2} - \frac{8}{3}\right) + 24U_2U_3\left(\frac{3\pi}{4}\right) \quad \ldots\ldots (21)$$

Or, $$\alpha = -4\pi\left(\frac{7}{6} - \frac{\beta}{4}\right)^2 - (54\pi)\left(\frac{\pi}{12} + \frac{\alpha}{12}\right)^2 - 24\left(\frac{\pi^2}{4} + \frac{4}{3}\right)^2\left(\frac{7}{6} - \frac{\beta}{4}\right)\left(\frac{\pi}{12} + \frac{\alpha}{12}\right) \quad \ldots (22)$$

$$and\ \beta = \left(\frac{32}{3}\right)\left(\frac{7}{6} - \frac{\beta}{4}\right)^2 + 36\left(\frac{\pi^2}{2} - \frac{8}{2}\right)\left(\frac{\pi}{12} + \frac{\alpha}{12}\right)^2 + 24\left(\frac{3\pi}{4}\right)\left(\frac{7}{6} - \frac{\beta}{4}\right)\left(\frac{\pi}{12} + \frac{\alpha}{12}\right) \quad \ldots (23)$$

From the coupled equations(22) and (23) we get the values of α and β (Using MATLAB program) as$\alpha = -\pi,\ \beta = \frac{8}{3}$; *and which give us,* $U_2 = \frac{7}{6} - \frac{\beta}{4} \Rightarrow U_2 = \frac{1}{2}$ and $U_3 = \frac{\pi+\alpha}{12} = \frac{\pi-\pi}{12} = 0$

Then from (19) , $\boldsymbol{u(x) = 1 - cos(x)}$, **is the exact (closed form) solution of the nonlinear integro-differential equation (13).**

6.Conclusion:

We have used two different decomposition methods (ADM and DTM) for finding solution of a nonlinear integro-differential equation (7). For ADM, it is observed that if we can adjust properly the values of first few terms ($u_0(x),\ u_1(x)\ etc.$) *then we can reach very near to the axact solution in term of series solution; and even to the exact solution as shown in our case.*

In case of DTM ,we have reached to a coupled nonlinear equations (22) and (23); and these equations are solved graphically by MATLAB for finding values of α *and* β. *By DTM we also have achieved the exact solution. If we are not able to reach to the closed form solution, in any case, then we can compare the series solutions obtained by ADM and DTM, numerically using MATLAB code.*

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